***Homework***

Read pages 11 – 14 in our textbook.

Do the following exercises: p. 14 #2, 3, 5, p. 15 #7, 8, 9

# Section 1.3 — Proof in Geometry

In Section 1.2, we illustrated some important ideas about the construction of a proof. We emphasized the following points:

1. A proof is based on assumptions and facts that we accept as true or that we have proven to be true.
2. Using these facts we can establish new conclusions, which we call theorems.

Now we'll continue our discussion of proof, in the context of geometry. Since we require a starting point, we begin by assuming three basic facts (or properties) to be true. As we proceed, you will realize that these properties are not difficult to prove. However, at this point they will be accepted as being true because they have been used in your past study of geometry and have become statements of fact that can be accepted as being true. They serve as a convenient starting point in our discussion of geometric proof. The proof of a property is called a theorem.

**Angles in a Triangle** **The sum of the interior angles in a triangle is 180°.**
**Isosceles Triangle Property** **In any isosceles triangle, the base angles are equal.**
**Opposite Angles** **When two straight lines intersect, opposite angles are equal.**

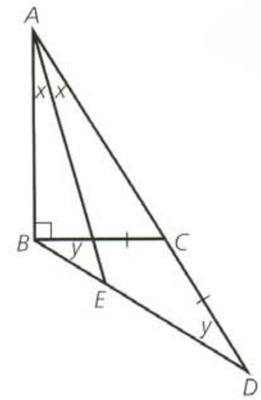
When we use theorems, we do not have to verify their correctness each time we wish to use them. The most important characteristic of theorems is that they are general in nature. For example, when we say, *The sum of the interior angles in a triangle is 180°*, we are describing every triangle. Given any triangle, we can be certain that its three angles will always add to this constant sum.

In geometric proofs, a diagram is imperative. On it we indicate all given information, and as we proceed we mark other facts as they are discovered.

EXAMPLE 1

Triangle $ABC$ has a right angle at $B$. $AC$ is extended to $D$ so that $CD = CB$. The bisector of angle $A$ meets $BD$ at $E$. Prove that $\angle AEB = 45^\circ$.

### Comment




We first draw a diagram, marking given information. Since $\angle A$ is bisected, we let $\angle BAE$ and $\angle EAD$ be $x$. Then $\angle BAC = 2x$. Because $CB = CD$, $\angle CDB = \angle CBD$, and they are each $y$. This is supplemental information that may be of use to us as we proceed. Note that we use the symbol $\angle CDB$ both as a name of the angle and to represent the measure of the angle.

### Proof

In $\Delta BCD$, $CB = CD$, so $\angle CDB = \angle CBD$.

Since the sum of the angles in a triangle is 180° in $\Delta ABD$, $(2x) + (y) + (90 + y) = 180$.

Then $$2x + 2y = 90$$
$$x + y = 45$$
In $\Delta ABE$, $x + y + 90 + \angle AEB = 180$
Then $$45 + 90 + \angle AEB = 180$$
$$\angle AEB = 45$$

Hence $$\angle \text{AEB} = 45^\circ.$$

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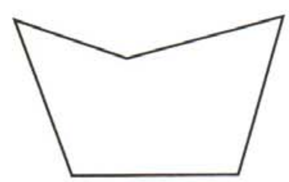
**EXAMPLE 2** Prove that for any convex polygon of $n$ sides, the sum of the interior angles is $180(n-2)^\circ$.

### Comment

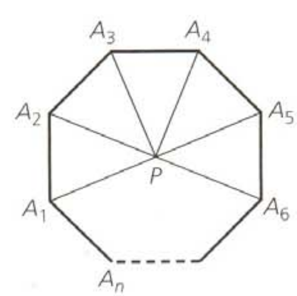
First we clarify the term *convex*. A convex polygon is a polygon in which each of the interior angles is less than 180°.

A convex polygon



A non-convex polygon

### Proof




Let $P$ be any point inside the polygon. Join $P$ to each of the vertices. (This is why we specified a convex polygon. There are points inside a non-convex polygon that cannot be joined to all vertices by lines that lie completely inside the polygon.) There are $n$ triangles, each having an angle sum of 180°, so the sum of the angles in the $n$ triangles is $180n^\circ$. The sum of the $n$ angles at $P$ is 360°, a complete rotation. Then the sum of the angles at the vertices is $180n^\circ - 360^\circ$, or $180(n-2)^\circ$.

In the first example, we dealt with a specific situation, while in the second we proved a property of convex polygons in general. Since this property has general applicability, we can consider it to be a new theorem and remember it for future use.

**Angles in a Convex Polygon Theorem** **In any convex polygon with $n$ sides (or vertices), the sum of the interior angles is equal to $180(n-2)^\circ$.**

In a convex polygon, every interior angle has an exterior angle associated with it. If a side of the polygon is extended, the exterior angle is the angle between the extended and adjacent sides. There are two exterior angles associated with each vertex because there are two sides that can be extended. It is easy to see that these two angles are equal, because they are formed by intersecting lines. In considering the exterior angles of a polygon, we count only one angle at each vertex.

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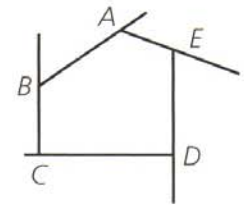
EXAMPLE 3

Prove that the sum of the exterior angles for any convex polygon is $360^\circ$.




### Proof

At any vertex, the sum of the interior and exterior angles is $180^\circ$. If there are $n$ vertices in the polygon, the sum of all interior and exterior angles is $180n^\circ$. The sum of the $n$ interior angles is $180(n-2)^\circ = (180n - 360)^\circ$. Then the sum of the exterior angles is $180n^\circ - (180n - 360)^\circ = 360^\circ$. Here we have another general result, one that is rather surprising. Regardless of the number of sides in a convex polygon, the sum of the exterior angles is $360^\circ$.

**Exterior Angles in a Convex Polygon Theorem** **The sum of the exterior angles of any convex polygon is $360^\circ$.**

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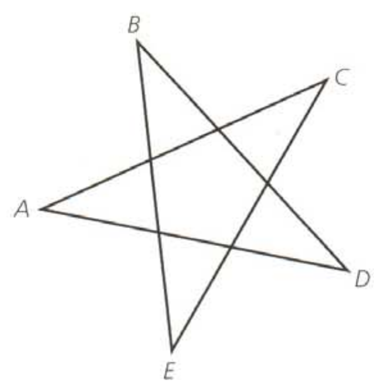
EXAMPLE 4

For the star-shaped figure shown, prove that the sum of the angles at $A$, $B$, $C$, $D$, and $E$ is $180^\circ$.




technology
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### Proof

Since vertically opposite angles formed by intersecting lines are equal, we indicate equal pairs in the diagram. In each of the triangles having one of the required angles, the angle sum is $180^\circ$.

$(\angle A + v + x) + (\angle B + x + y) + (\angle C + y + z) + (\angle D + z + u) +$
$(\angle E + u + v) = 5(180)$

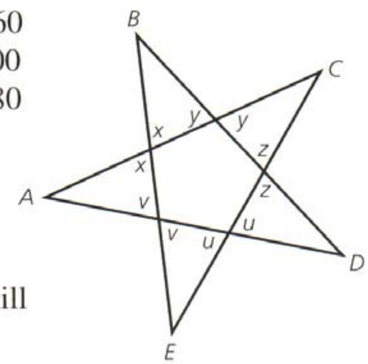
Then $\angle A + \angle B + \angle C + \angle D + \angle E + 2(x + y + z + u + v) = 5(180) = 900$

For the interior polygon, the sum of the exterior angles is 360°.

Then $x + y + z + u + v = 360$

Therefore $\angle A + \angle B + \angle C + \angle D + \angle E + 720 = 900$

$\angle A + \angle B + \angle C + \angle D + \angle E = 180$




The sum of the five angles is 180°.

In this section, we have tried to give you an idea of the nature of geometric proof and how to construct a proof. The exercises give you an opportunity to develop the skill of writing simple proofs.

## Exercise 1.3

### Part A

1. A convex polygon has 12 sides. Given that all interior angles are equal, prove that every angle is 150°.

2. In $\Delta KLM$, $P$ is the midpoint of the line segment $LM$. Prove that if $PL = PK = PM$, $\angle LKM = 90°$.

### Part B

Knowledge/Understanding

3. Prove that the sum of the exterior angles at opposite vertices of any quadrilateral is equal to the sum of the interior angles at the other two vertices.

4. In $\Delta ABC$, $\angle A$ is a right angle. The bisectors of $\angle B$ and $\angle C$ meet at $D$. Prove that $\angle BDC = 135°$.

Application

5. In $\Delta PQR$, $PQ = PR$. $PQ$ is extended to $S$ so that $QS = QR$. Prove that $\angle PRS = 3(\angle QSR)$.

6. The number of degrees in one interior angle of a regular polygon is $x°$. Prove that a formula for the number of sides of the polygon is $\frac{360}{180 - x}$.
(Remember that a regular polygon has equal sides and equal interior angles.)

7. $\Delta ABC$ is obtuse-angled at $C$. The bisectors of the exterior angles at $A$ and $B$ meet $BC$ and $AC$ extended at $D$ and $E$, respectively. If $AB = AD = BE$, prove that $\angle ACB = 108°$.

8. In $\Delta ABC$, the bisector of the interior angle at $A$ and the bisector of the exterior angle at $B$ intersect at $P$. Prove that $\angle APB = \frac{1}{2} \angle C$.

Thinking/Inquiry/ Problem Solving

9. a. For the following seven-pointed star, determine the sum of the angles at the tips of the star.

   b. If a star has $n$ points, where $n$ is an odd number, find a formula for the sum of the angles at the tips of the star.