Homework

pp. 19-20 #2, 4, 6, 7, 8, 9, 11, 12, 13

Section 1.4 — Proof With Analytic Geometry

In earlier grades, you used analytic geometry to solve problems. Since analytic geometry combines geometric properties with algebraic methods, it is useful in numeric problems and in problems requiring general proofs.

EXAMPLE 1

Determine the coordinates of the point that is equidistant from the three points $A(-2, 2)$, $B(6, 10)$, and $C(12, -2)$.

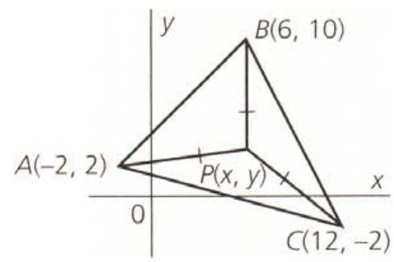
Solution

Let $P(x, y)$ be the required point.

Then $PA = PB = PC$

or

$$PA^2 = PB^2 = PC^2$$



$$PA^2 = (x + 2)^2 + (y - 2)^2 = x^2 + 4x + y^2 - 4y + 8 \quad ①$$
$$PB^2 = (x - 6)^2 + (y - 10)^2 = x^2 - 12x + y^2 - 20y + 136 \quad ②$$
$$PC^2 = (x - 12)^2 + (y + 2)^2 = x^2 - 24x + y^2 + 4y + 148 \quad ③$$

Equating ① and ②, $x^2 + 4x + y^2 - 4y + 8 = x^2 - 12x + y^2 - 20y + 136$

$$16x + 16y = 128$$
$$x + y = 8 \quad ④$$

Equating ① and ③, $x^2 + 4x + y^2 - 4y + 8 = x^2 - 24x + y^2 + 4y + 148$

$$28x - 8y = 140$$
$$7x - 2y = 35 \quad ⑤$$

Multiply equation ④ by 2: $2x + 2y = 16$ ⑥

Adding ⑤ and ⑥ $9x = 51$

$$x = \frac{17}{3}$$

Then $y = 8 - x$

$$= 8 - \frac{17}{3}$$
$$= \frac{7}{3}$$

The point $P\left(\frac{17}{3}, \frac{7}{3}\right)$ is equidistant from A, B, and C.

Using these same techniques, we can consider general proofs. Since we wish them to be general, we cannot use numeric values for points defining a figure. Instead, we name the coordinates of points in general terms. We begin by listing the basic facts with which you are familiar.

You know

1. how to determine the distance between two points
2. how to determine the equation of a line given its slope and a point on the line
3. how to determine the equation of a line given two points on the line
4. that two lines are parallel if their slopes are equal
5. that two lines are perpendicular if their slopes are negative reciprocals or if the product of the slopes is –1
6. how to determine the equation of a circle given its radius and centre

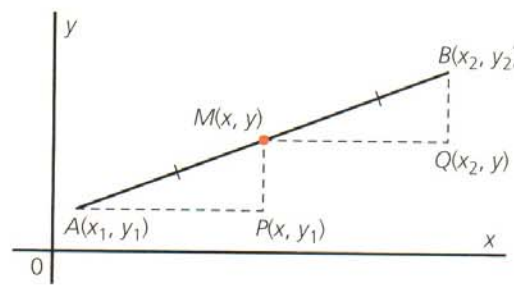
We can use these facts to consider the following examples.

EXAMPLE 2

Prove that the midpoint of the line segment connecting the points $A(x_1, y_1)$ and $B(x_2, y_2)$ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Proof

Let the midpoint of the line segment be $M(x, y)$. Draw the run and the rise from A to M and from M to B. Then P has coordinates (x, y_1) and Q has coordinates (x_2, y).



In ΔAPM and ΔMQB, $AM = MB$

$\angle APM = \angle MQB$ (Right angle)

$\angle MAP = \angle BMQ$ (Lines parallel)

ΔAPM is congruent to ΔMQB.

Then $AP = MQ$ and $PM = QB$

Then $x - x_1 = x_2 - x$ and $y - y_1 = y_2 - y$

or $2x = x_1 + x_2$ and $2y = y_1 + y_2$

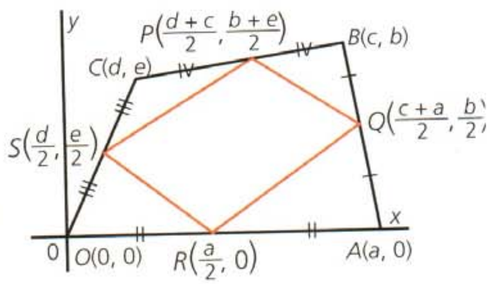
or $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$

The coordinates of M are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

The midpoint of the line segment connecting points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

EXAMPLE 3

Prove that the lines joining consecutive midpoints of the sides of a convex quadrilateral form a parallelogram.



Proof

Let the coordinates for the quadrilateral be $O(0, 0)$, $A(a, 0)$, $B(c, b)$, and $C(d, e)$. (Note that we have placed one vertex at $(0, 0)$ and one side along the x-axis. This will simplify calculations.)

Since we wish to show that the midpoints of the quadrilateral form a parallelogram, we first calculate the coordinates for the midpoints of the four sides. The midpoint of OA is $R\left(\frac{a}{2}, 0\right)$.

Similarly, the midpoints of OC, BA, and CB are $S\left(\frac{d}{2}, \frac{e}{2}\right)$, $Q\left(\frac{c+a}{2}, \frac{b}{2}\right)$, and $P\left(\frac{d+c}{2}, \frac{b+e}{2}\right)$, respectively.

Now the slope of SP is $\dfrac{\frac{b+e}{2} - \frac{e}{2}}{\frac{d+c}{2} - \frac{d}{2}} = \dfrac{b}{c}$ and the slope of RQ is $\dfrac{\frac{b}{2} - 0}{\frac{c+a}{2} - \frac{a}{2}} = \dfrac{b}{c}$

The slopes are equal, so SP is parallel to RQ.

Also, the slope of RS is $\dfrac{\frac{e}{2} - 0}{\frac{d}{2} - \frac{a}{2}} = \dfrac{e}{d-a}$

and the slope of QP is $\dfrac{\frac{b+e}{2} - \frac{b}{2}}{\frac{d+c}{2} - \frac{c+a}{2}} = \dfrac{e}{d-a}$

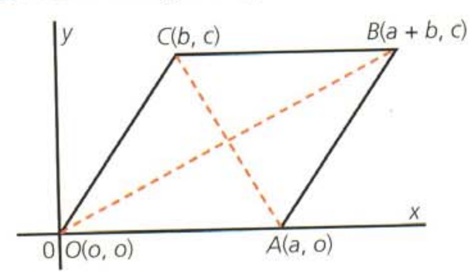
Then RS is parallel to QP.

Since the opposite sides are parallel, $PQRS$ is a parallelogram.

In Example 3, it was noted that convenient placing of a figure relative to the axes can simplify calculations. We must also be aware that geometric conditions will impose conditions on coordinates. This is important in the next example.

EXAMPLE 4

Prove that the diagonals of a rhombus bisect each other at right angles.



Proof

Let the rhombus have coordinates $O(0, 0)$, $A(a, o)$, $C(b, c)$, and $B(a + b, c)$. (This makes $CB = OA$.)
Since $OABC$ is a rhombus, $OA = OC$.
Then $a^2 = b^2 + c^2$, so $c^2 = a^2 - b^2$.

Now, slope $OB = \frac{c}{a+b}$ and slope $AC = \frac{c}{b-a}$, so the product of the slopes is $\frac{c}{a+b} \times \frac{c}{b-a} = \frac{c^2}{b^2 - a^2}$.

But $c^2 = a^2 - b^2$, so the product is $\frac{a^2 - b^2}{b^2 - a^2} = -1$,

and the diagonals are perpendicular.

The midpoint of OB is $P\left(\frac{a+b}{2}, \frac{c}{2}\right)$. The midpoint of AC is $Q\left(\frac{a+b}{2}, \frac{c}{2}\right)$.

Since these coordinates are the same, P and Q are the same point. The midpoint of one diagonal is also the midpoint of the other. Then the diagonals of a rhombus bisect each other at right angles.

Exercise 1.4

Part A

Always try to choose coordinates of points so as to simplify your work. It can sometimes be helpful to work a specific example before doing the general case.

1. Prove that the diagonals of a parallelogram bisect each other.

Knowledge/
nderstanding

2. Triangle ABC has vertices $A(-1, 3)$, $B(5, 5)$, and $C(7, -1)$.
   a. Prove that this triangle is isosceles.
   b. Prove that the line through $B(5, 5)$ perpendicular to AC passes through the midpoint of AC.

Part B

3. The ΔXYZ has its vertices at $X(5, 4)$, $Y(-2, 2)$, and $Z(9, -3)$.
   a. Determine the equation of the line drawn from vertex X to the midpoint of YZ. (The line in a triangle from a vertex to the midpoint of the opposite side is called a **median**.)
   b. Determine the equation of the median drawn from vertex Y.
   c. Prove that the two medians from parts **a** and **b** intersect at the point (4, 1).
   d. Verify that the point (4, 1) lies on the median drawn from vertex Z.
   e. What conclusion can be drawn about the medians of this triangle?

4. The ΔPQR has its vertices at $P(-6, 0)$, $Q(0, 8)$, and $R(4, 0)$.
   a. Determine the equation of a line drawn from R that is perpendicular to PQ. (This line is called the **altitude** from R to PQ.)

b. Determine the equation of the altitude drawn from P to QR.

c. Determine the coordinates of the point of intersection of the two altitudes found in parts **a** and **b**.

d. Show that the altitude from Q contains the point of intersection of the other two altitudes.

e. What conclusion can be drawn about the altitudes of this triangle?

Application

5. Prove that the diagonals in a square are equal.

6. Prove that the diagonals in a rectangle are equal.

Application

7. Prove that the line segments joining the midpoints of any rectangle form a rhombus.

8. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and has a length equal to one half of it.

9. Prove that the sum of the squares of the lengths of the sides of a parallelogram is equal to the sum of the squares of the lengths of the diagonals.

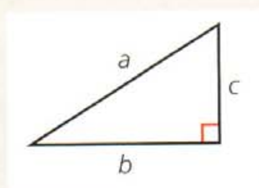
10. In ΔABC, where AD is the median, prove that $AB^2 + AC^2 = 2BD^2 + 2AD^2$. (Let the length of BC be $2a$ units and use B or D as the origin.)

11. a. In the interior of rectangle $ABCD$, a point P is chosen at random. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

b. Prove this result using the Pythagorean Theorem.

Pythagorean Theorem

If a triangle is right-angled, the square on the hypotenuse is equal to the sum of the squares on the other two sides; that is, $a^2 = b^2 + c^2$.



Part C

Thinking/Inquiry/ Problem Solving

12. Prove that the altitudes of a triangle are concurrent.

13. Prove that the medians of a triangle are concurrent.