Unit 2 – Geometric and Algebraic Vectors and their Applications

Vectors and Scalars

Definition

Quantities having **magnitude only** are called ***scalars*****.**
Quantities having both **magnitude and direction** are called ***geometric vectors***. (For the sake of simplicity, we usually omit the modifier "geometric" and simply call them ***vectors***. However, there is an important distinction between geometric vectors and ***algebraic vectors*** that will be made clear later in this unit.)

Examples

Scalars	***Vectors***
Temperature	Wind Velocity
Energy	Position
Distance	Displacement
Speed	Velocity

Exercises

1. What is the difference between speed and velocity? What is the difference between distance and displacement?

2. Is it possible for speed to increase while velocity decreases?

3. Geometric vectors can be one-dimensional, two-dimensional, three-dimensional or even of higher dimension (if you are imaginative). Is there any difference between a scalar and a one-dimensional vector?

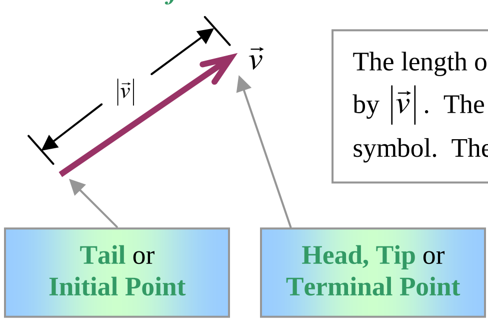
4. Classify each of the following quantities as scalars or vectors.

acceleration, magnetic field, electric charge, force, mass, area, time, volume, density, pressure of a gas

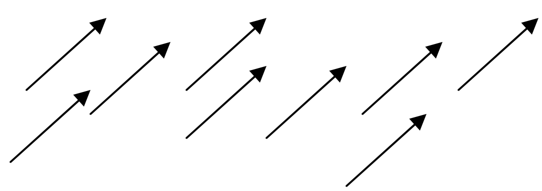
Concepts and Notation

Geometric Vector

A geometric vector is represented by an arrow pointing in a certain direction. The **length** of the arrow is called the ***magnitude*** of the vector and the **direction** in which it points is called its ***direction***. A geometric vector is entirely determined by its magnitude and its direction. Consequently, the position of a vector in space is unimportant. Such vectors are called ***free vectors***.

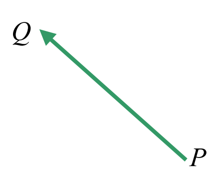


The length of the vector is its magnitude. The magnitude of the vector $\vec{v}$ is denoted by $|\vec{v}|$. The symbol used for magnitude should remind you of the ***absolute value*** symbol. The absolute value of a scalar is its magnitude.



All the vectors shown at the left are equal because they all have the same magnitude and the same direction

We can also use the endpoints of a line segment to name vectors. The vector $\overrightarrow{PQ}$ is shown below. P is the initial point and Q is the terminal point. We can think of the ***directed line segment PQ***, which is fixed in space, as a single copy of the vector $\overrightarrow{PQ}$.



Defining Vector Operations

The Process of Defining Mathematical Objects

Often, the definition of mathematical objects seems to be confusing and arbitrary. A classic example of this is the definition of powers with zero and negative exponents. Most students are perplexed and dumbstruck when they are told that $10^0 = 1$ and that $10^{-n} = \frac{1}{10^n}$. However, if we would like our laws of exponents to hold and we would like exponential curves to be smooth and to model natural processes (such as population growth and exponential decay) accurately, then we are forced to accept these definitions!

Why is $10^0 = 1$?

Let $x \in \mathbb{R}$.

$$10^0 = 10^{x-x} = \frac{10^x}{10^x} = 1$$

$$\therefore 10^0 = 1$$

Why is $10^{-n} = \frac{1}{10^n}$?

Let $x \in \mathbb{R}$.

$$10^{-n} = 10^{x-(x+n)} = \frac{10^x}{10^{x+n}} = \frac{10^x}{10^x 10^n} = \frac{1}{10^n}$$

$$\therefore 10^{-n} = \frac{1}{10^n}$$

Problems that Motivate the Definitions of Vector Addition and Multiplication of a Vector by a Scalar

Problem # 1

Ryanna and Philip need to move a large boulder from the middle of their campsite. Ryanna pushes due North and Philip pushes due West. If $\vec{R}$ represents the force with which Ryanna pushes, $\vec{P}$ represents the force with which Philip pushes and $|\vec{R}| = 2|\vec{P}|$, in which direction will the boulder move? **Note:** $|\vec{R}| = 2|\vec{P}|$ means that the magnitude of Ryanna's force is twice the magnitude of Philip's force.

Problem # 2

Once Ryanna and Philip finished moving the boulder, Snehjot came along and shrieked, "You should have moved the boulder 10° further to the west, you stupid, asinine, mindless, vacuous fools!" Insulted by Snehjot's strong language, Philip refused to change the way he was pushing the boulder. Describe how Ryanna would have to adjust the magnitude (but not direction) of his/her force to move the boulder 10° further to the West.

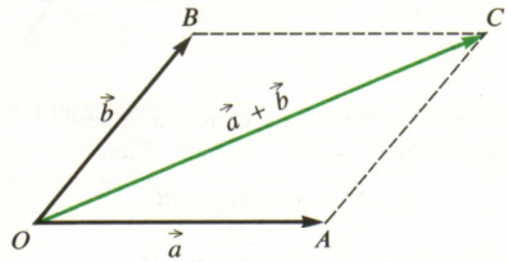
The Definitions of Vector Addition and Subtraction

Any solution to problem 1 above requires the concept of adding two vectors:

Parallelogram Law of Addition

Let $\overrightarrow{OA}$ represent the vector $\vec{a}$ and let $\overrightarrow{OB}$ represent $\vec{b}$, so that the vectors start at the same point. Complete the parallelogram $OACB$. The sum $\vec{a} + \vec{b}$ is represented by the diagonal $\overrightarrow{OC}$. Hence,

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$



The sum $\vec{a} + \vec{b}$ is also called the **resultant** of $\vec{a}$ and $\vec{b}$. Notice that the magnitude, or length, of $\vec{a} + \vec{b}$ is less than or equal to the combined magnitudes of $\vec{a}$ and $\vec{b}$; that is,

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

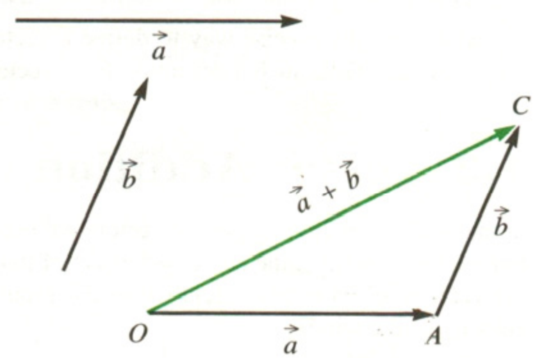
Therefore, when two people try to push the rock as in the example, the effect will be less than the sum of the individual effects. If however, they were to push in the same direction, the magnitude of the resultant would be equal to the sum of the two magnitudes.

In general, if we wish to add the vectors $\vec{a}$ and $\vec{b}$, we have to choose representative line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ that start at the same point O. Note, that in the parallelogram $OACB$, $\vec{b} = \overrightarrow{OB} = \overrightarrow{AC}$, so that the sum of $\vec{a}$ and $\vec{b}$ can also be computed by using the triangle OAC.

Triangle Law of Addition

If $\vec{a}$ and $\vec{b}$ are vectors, choose points O and A so that $\vec{a} = \overrightarrow{OA}$. Then find the point C so that $\vec{b} = \overrightarrow{AC}$; the vectors are now arranged "head to tail." The sum $\vec{a} + \vec{b}$ is represented by the side $\overrightarrow{OC}$. Hence,

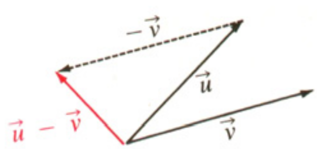
$$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$



The Triangle Law can be thought of as follows. Suppose you walk from O to A; you have then moved along the vector $\vec{a} = \overrightarrow{OA}$. If you now walk from A to C, you have moved along $\vec{b} = \overrightarrow{AC}$; your final position would have been the same if you would have walked along the vector $\vec{a} + \vec{b} = \overrightarrow{OC}$.

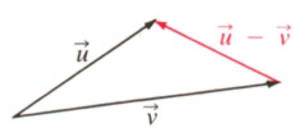
Should I use the Parallelogram Law or the Triangle Law?

The answer to this question is that it doesn't matter! Both laws produce the same resultant. If your diagram has two vectors arranged tail-to-tail, then it's easier to use the parallelogram law. If the vectors are arranged head-to-tail, then it's easier to use the triangle law. In the final analysis, however, it's just a matter of personal preference.



For vectors $\vec{u}$ and $\vec{v}$, the **difference**, $\vec{u} - \vec{v}$, is $\vec{u} + (-\vec{v})$.

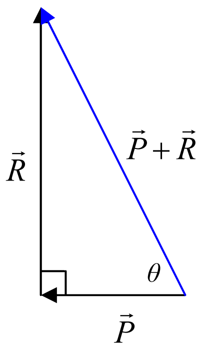
Note that $\vec{u} - \vec{v}$ can also be illustrated as in the following diagram.



Multiplying a Vector by a Scalar

To solve problem 2, we require the concept of multiplying a vector by a scalar. Let's examine a solution to help us understand how this form of multiplication works.

Solution to Problem 1



$$\tan\theta = \frac{|\vec{R}|}{|\vec{P}|}$$

$$= \frac{2|\vec{P}|}{|\vec{P}|}$$

$$=2$$

Therefore,

$\theta = \tan^{-1} 2 \doteq 63.4°$

The boulder would move N26.6°W.

Solution to Problem 2

What kind of answer should we expect?

As with every problem, we should think first about what a reasonable final answer would be. It's clear from the diagram that Ryanna should ***reduce*** the force with which he/she pushes. Therefore, we should expect the value of a to be less than 1.

Ali wanted the boulder to move approximately N36.6°W. Let a represent the scalar by which $\vec{R}$ needs to be multiplied so that the boulder will move in the desired direction. Therefore,

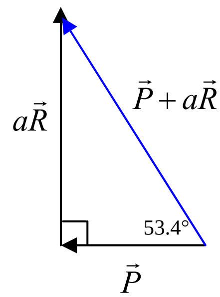
$$\frac{|a\vec{R}|}{|\vec{P}|} = \tan 53.4°$$

$$\therefore \frac{|a(2\vec{P})|}{|\vec{P}|} = \tan 53.4°$$

$$\therefore \frac{|2a||\vec{P}|}{|\vec{P}|} = \tan 53.4°$$

$$\therefore |2a| = \tan 53.4°$$

$$\therefore a \doteq 0.66$$



Ryanna should push in the same direction, with about two-thirds of the force with which he/she pushed initially.

Intuitive Understanding of Multiplying a Vector by a Scalar

Multiplying a vector by a scalar "stretches" the vector. Depending on the value of the scalar, the "stretched" vector can be longer or shorter than the original vector and can be in the same direction or the opposite direction of the original vector. Complete the following table to ensure that you understand all the possibilities.

Algebraic Expression	*Diagram*	*Explanation*
$\vec{u} = c\vec{v}, c > 1$		
$\vec{u} = c\vec{v}, 0 < c < 1$		
$\vec{u} = c\vec{v}, c < -1$		
$\vec{u} = c\vec{v}, -1 < c < 0$		

Unit Vectors and the Zero Vector

A ***unit vector*** is a vector with a magnitude of ***one*** unit. Despite what some students may believe, unit vectors do not have a special status in the world of vectors. The only thing that sets them apart from all other vectors is that it is particularly easy to work with them. Unit vectors are often used to specify a ***direction***.

How to find a Unit Vector in the Direction of a Given Vector

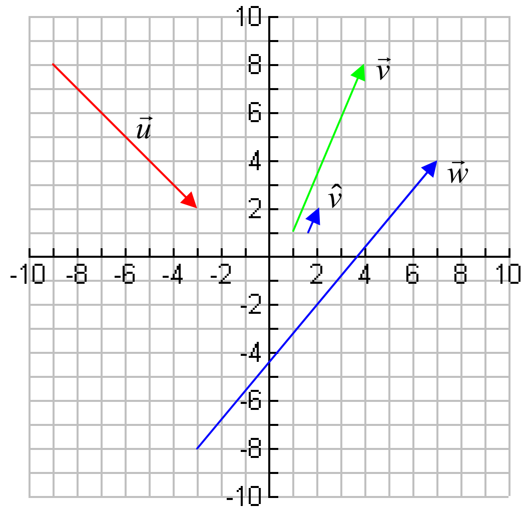
Given the vector $\vec{v}$, the vector $\hat{v} = \frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}$ is a unit vector in the direction of $\vec{v}$.

Any vector $\vec{v}$ can be expressed in terms of a unit vector in the direction of $\vec{v}$:

$$\vec{v} = |\vec{v}|\,\hat{v}$$

Example and Exercises

For each vector in the diagram, find a unit vector in the direction of the given vector. In addition, use the diagram at the left to sketch the unit vector.



Solution

Using the Pythagorean Theorem,

$$|\vec{v}|^2 = 3^2 + 7^2$$
$$\therefore |\vec{v}|^2 = 58$$
$$\therefore |\vec{v}| = \sqrt{58}$$
$$\therefore \hat{v} = \tfrac{1}{\sqrt{58}}\vec{v}$$

The remaining exercises are left up to you.

The Zero Vector

Although the ***zero vector*** is as simple as a vector can be, it can be a little difficult to understand from a geometric perspective. The zero vector, denoted by $\vec{0}$, has length (magnitude) zero and has indeterminate direction.

Applications of the Zero Vector

- If several forces act on an object but it remains stationary, the resultant force on the object is zero. That is, the ***net force*** on the object is $\vec{0}$.
- If a journey ends exactly where it began, the ***displacement*** is $\vec{0}$.
- List as many other applications as you can of the zero vector:

Properties of Vectors

Instructions

The following table lists many important properties of vectors. Complete the table with a diagram and an explanation for each property.

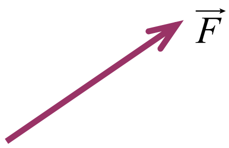
Property	*Diagram*	*Explanation*
$\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (Commutative Law)		
$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ (Associative Law)		
$(ab)\vec{u}=a(b\vec{u})$ (Associative Law)		
$a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$ (Distributive Law)		
$\vec{u}+\vec{0}=\vec{u}$ (Existence of Additive Identity)		
$\vec{u}+(-\vec{u})=\vec{0}$ (Existence of Additive Inverse)		
$\lvert c\vec{u}\rvert=\lvert c\rvert\lvert\vec{u}\rvert$		
$\lvert\vec{u}+\vec{v}\rvert\le\lvert\vec{u}\rvert+\lvert\vec{v}\rvert$ $\lvert\vec{u}-\vec{v}\rvert\le\lvert\vec{u}\rvert+\lvert\vec{v}\rvert$ $\lvert\vec{u}+\vec{v}\rvert\ge\lvert\lvert\vec{u}\rvert-\lvert\vec{v}\rvert\rvert$ (Triangle Inequalities)		

Check Your Understanding of Basic Vector Concepts

Statement	*True or False?*	*Proof or Counterexample*
Two vectors are parallel if and only if one is a scalar multiple of the other. ($\vec{u} \parallel \vec{v}$ iff $\vec{u} = k\vec{v}$ for some $k \in \mathbb{R}$)		
$-\overrightarrow{PQ} = \overrightarrow{QP}$		
$\lvert\vec{u}+\vec{v}\rvert = \lvert\vec{u}\rvert + \lvert\vec{v}\rvert$ iff $\vec{u} = k\vec{v}$ for some $k \in \mathbb{R}$		
$\lvert\vec{u}+\vec{v}\rvert < \lvert\vec{u}\rvert + \lvert\vec{v}\rvert$ iff $\vec{u} \neq k\vec{v}$ for any $k \in \mathbb{R}$		
Since for all vectors $\vec{u}$ and all scalars *c*, $\lvert c\vec{u}\rvert = \lvert c\rvert\lvert\vec{u}\rvert$, it must follow that $\vec{u}$ and $c\vec{u}$ have the same direction.		

Using Vectors to Represent Forces and Velocities

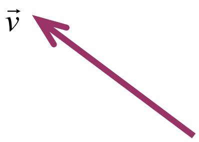
Force



If a vector is used to represent a force,

1. The ***direction*** of the vector represents the ***direction in which the force is applied***. This is usually measured in radians, degrees or by using the compass directions.
2. The ***magnitude*** of the vector represents the ***"strength"*** of the force, which is usually measured in Newtons (N = kg·m/s^2). One Newton (1 N) is the ***resultant force*** that will give a 1 kg mass an acceleration of 1 m/s^2, that is, 1 (m/s)/s.

Velocity

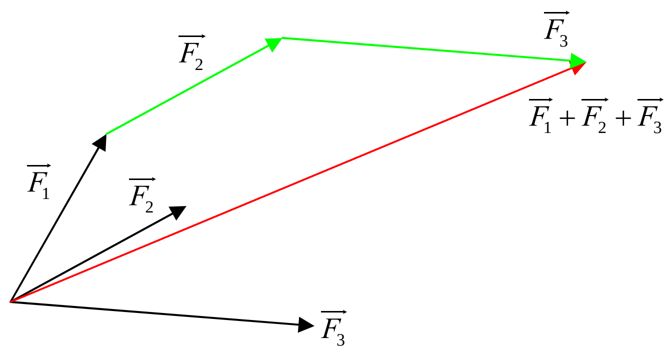


If a vector is used to represent a velocity,

1. The ***direction*** of the vector represents the ***direction of movement***. This is usually measured in radians, degrees or by using the compass directions.
2. The ***magnitude*** of the vector represents the ***speed*** of the moving object, which is usually measured in m/s or km/h.

Note

1. If several forces act on an object and their lines of action all pass through a common point (such forces are called ***concurrent***), the ***resultant force vector*** (sum of all the force vectors) represents the ***combined effect*** of all the forces.



2. In the above diagram, $\vec{F}_1 + \vec{F}_2 + \vec{F}_3$ is called the ***resultant force***.

3. In the above diagram, $-(\vec{F}_1 + \vec{F}_2 + \vec{F}_3)$ is called the ***equilibrant force*** (the force that exactly counterbalances the resultant).

Relative Velocity

Velocity is always ***relative*** to the ***frame of reference*** of the observer. This means that different observers in different frames of reference will measure different velocities! The example on the next page uses one-dimensional motion (motion along a straight line) to illustrate clearly the relativity of motion.

One-Dimensional Motion

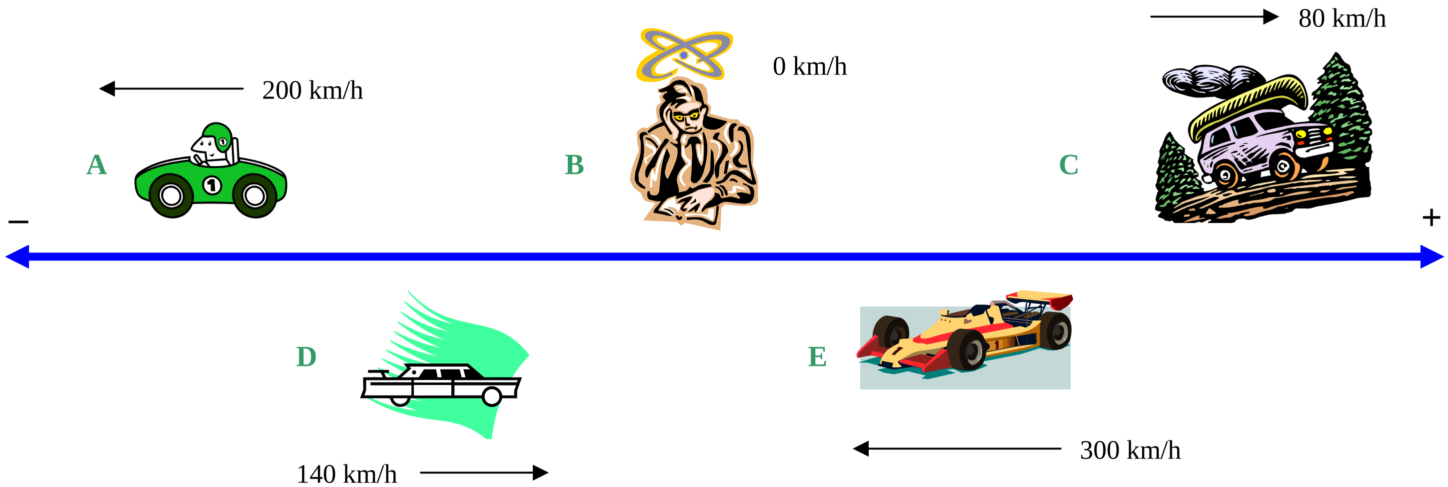
In ***one-dimensional motion***, only ***two*** directions are possible. For the sake of convenience, one of these directions is called the ***positive direction*** and the other direction is called the ***negative direction*****.** The chart shown below summarizes some common interpretations of these two directions.

Negative Direction	*Positive Direction*
Left	Right
Down	Up
South	North
West	East
N45°W	S45°E

While the above table summarizes some common ways of interpreting the positive direction and the negative direction, it is by no means exhaustive. In fact, there are an infinite number of possible interpretations because all that is required is that the two directions be opposite each other.

In one-dimensional motion, velocities are specified by using a real number. For instance, a velocity of –100 km/h means that the ***speed*** (the magnitude) is 100 km/h and the ***direction*** is the negative direction.

Example Involving One-Dimensional Motion



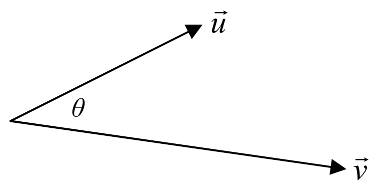
Now complete the following table. (The abbreviation “wrt” stands for “with respect to,” which means the same thing as “relative to.”) Can you draw any general conclusions about relative velocities?

Velocity of	Relative to (wrt)	Relative Velocity (km/h)	Velocity of	Relative to (wrt)	Relative Velocity (km/h)	Velocity of	Relative to (wrt)	Relative Velocity (km/h)	Velocity of	Relative to (wrt)	Relative Velocity (km/h)	Velocity of	Relative to (wrt)	Relative Velocity (km/h)
A	A	0	B	A		C	A		D	A		E	A	
A	B	–200	B	B		C	B		D	B		E	B	
A	C	–280	B	C		C	C		D	C		E	C	
A	D	–340	B	D		C	D		D	D		E	D	
A	E	100	B	E		C	E		D	E		E	E	

Note: The method that you are using to calculate relative velocities is based on what are known as the Galilean transformations. These transformations work very well for velocities much smaller than the speed of light ($c = 3\times10^8$ m/s). For velocities approaching c, however, the Galilean transformations break down and must be replaced by the Lorentz transformations, which form the foundation of Einstein’s Special Theory of Relativity.

The Angle between Two Vectors and Relative Velocities

The angle between two vectors is the ***non-reflex angle θ*** (i.e. $0° \le \theta \le 180°$) obtained when they are arranged tail-to-tail.

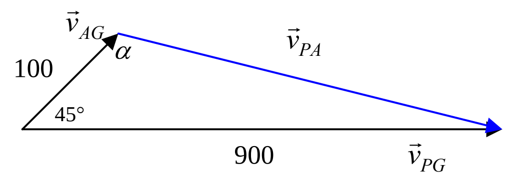


Example

A Boeing 747 travels due east in still air (no wind) at a speed of 900 km/h. It suddenly encounters a 100 km/h SW wind (i.e. blowing ***from*** the SW, which means the wind is moving NE). How should the pilot adjust the ***heading*** of the aircraft to ensure that it continues to move due east at 900 km/h?

Solution

We want the ***resultant*** velocity to be 900 km/h, due east. This means that the ***combined effect*** of the wind and the direction in which the pilot steers should cause the aircraft to move due east with a speed of 900 km/h. If we let $\vec{v}_{PG}$ represent the velocity of the plane relative to the ground, $\vec{v}_{PA}$ represent the velocity of the plane relative to the air and let $\vec{v}_{AG}$ represent the velocity of the air relative to the ground (i.e. wind velocity). Then, we require that



$$\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG} \text{ or } \vec{v}_{PA} = \vec{v}_{PG} - \vec{v}_{AG}.$$

By the law of cosines,

$$|\vec{v}_{PA}|^2 = |\vec{v}_{AG}|^2 + |\vec{v}_{PG}|^2 - 2|\vec{v}_{AG}||\vec{v}_{PG}|\cos\theta$$, where θ is the angle between $\vec{v}_{AG}$ and $\vec{v}_{PG}$.

$$\therefore |\vec{v}_{PA}|^2 = 100^2 + 900^2 - 2(100)(900)\cos 45°$$

$$\therefore |\vec{v}_{PA}| \doteq 832$$

By the law of sines,

$$\frac{\sin\alpha}{|\vec{v}_{PG}|} = \frac{\sin\theta}{|\vec{v}_{PA}|}$$

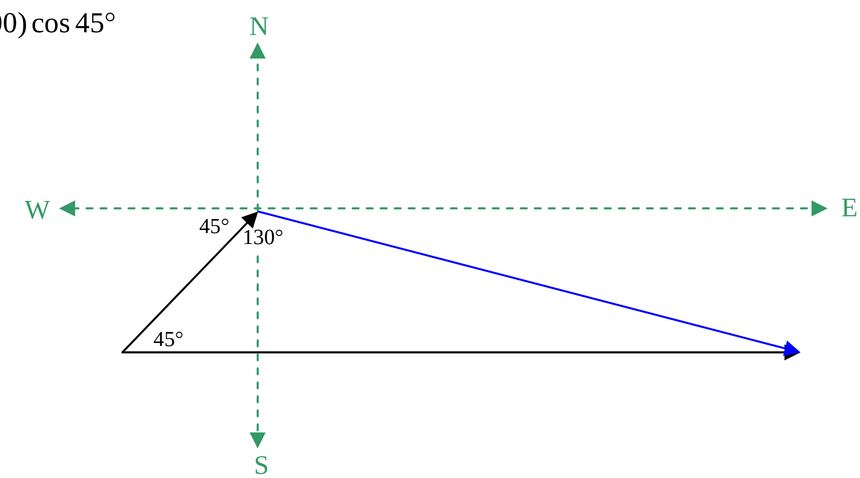
$$\therefore \frac{\sin\alpha}{900} \doteq \frac{\sin 45°}{832}$$

$$\therefore \sin\alpha \doteq \frac{900\sin 45°}{832}$$

$$\therefore \alpha \doteq \sin^{-1}\left(\frac{900\sin 45°}{832}\right)$$

$$\therefore \alpha \doteq \sin^{-1}\left(\frac{900\sin 45°}{832}\right)$$

~~$\alpha \doteq 50°$~~ or $\alpha \doteq 130°$



Beware! ***This is the ambiguous case of the sine law***. The sine of an angle is positive in both the first quadrant and the second quadrant. Therefore, we must choose our answer carefully!

To continue along the desired course, the pilot must head S85°E with an air speed of about 832 km/h.

Questions

1. Explain why α must have a value of 130° and not 50°.
2. How is it possible for the pilot to reduce the air speed from 900 km/h to 832 km/h and still maintain a ground speed of 900 km/h?
3. Explain how the direction S85°E was obtained. Use the given diagram and relevant theorems of geometry.

Examples

1. Describe the forces acting on an aircraft flying at a constant speed at a constant altitude.
2. Describe the forces acting on a submarine cruising at a constant speed at a constant depth.
3. Describe the forces acting on an automobile moving at a constant speed on a flat horizontal road.
4. A force of 200 N is being applied to a rope to pull a toboggan along a horizontal, frictionless surface. If the rope forms an angle of 60° to the horizontal, find the horizontal and vertical components of the force (this is called ***resolving*** the vector). Which of the components does all the work, the vertical or the horizontal? What does this tell you about the angle at which the rope should be pulled? Should the angle be as close to 90° as possible or as close to 0° as possible?
5. A 100 kg object ***rests*** on a ramp inclined at a certain angle θ to the horizontal. Calculate the components of the force of gravity on the object that are parallel and perpendicular to the ramp. Express your answers in terms of θ.
6. Suppose now that the angle θ (in question 5) is increased slowly until a critical angle θ_c is reached and the object begins to slide down the ramp. What can you conclude about the force of friction on the object when this critical angle is reached?
7. Two draft horses pull a load. The chains between the horses and the load are at an angle of 60° to each other. One horse pulls with a force of 230 N and the other pulls with a force of 340 N. What is the resultant force on the load? What is the equilibrant force on the load? (State both magnitude and direction.)
8. A traffic sign with a mass of 5 kg is suspended above a street by two cables. One cable forms an angle of 45° to the street and the other forms an angle of 60°. Find the tension of each wire.
9. An airplane is steering at N45°E with an air speed (speed in still air) of 525 km/h. The wind is from N60°W at 98 km/h. Find the groundspeed and track (course) of the airplane.
10. A ship is steering east at 15 knots (nautical miles per hour). A tugboat 2 nautical miles (M) to the south is steering N45°E at 20 knots. Find the velocity of the ship relative to the tug. Will the ship pass in front of or behind the tug?

To solve problems like the ones given above, it is critical that you have a good *understanding* of the *underlying physical principles*! Formulas are not enough!

Newton's Laws of Motion

Newton's First Law (Galileo's Law of Inertia)

Every body in a state of ***uniform motion*** tends to remain in that state of motion unless an ***external force*** is applied to it. In other words, unless an external force is applied to a body, if it is at rest, it will remain at rest, and if it is moving with a constant velocity, it will continue to move with a constant velocity. (The ***mass*** of a body is a measure of its ***inertia***, that is, a body's resistance to acceleration.)

Newton's Second Law

In the presence of a ***net (resultant) force***, a body experiences an acceleration that is directly proportional to the net force and inversely proportional to the mass. This law can be summarized using the equation

$\vec{a} = \dfrac{\vec{F}}{m}$ or $\vec{F} = m\vec{a}$. (Newton originally expressed this law using calculus: $\vec{F} = \dfrac{d\vec{p}}{dt}$, where $\vec{p}$ represents the momentum of the body.)

When a body is acted upon by a gravitational field, it experiences an acceleration called the ***acceleration due to gravity*** or ***gravitational acceleration***. Close to the surface of the Earth, this acceleration is ***roughly constant*** and is ***equal to approximately*** 9.8 m/s^2. The symbol g is used to represent acceleration due to gravity. Newton's second law informs us in this case that $F = mg$, where F represents the magnitude of the gravitational force exerted on a body of mass m. Note that gravitational forces are always directed toward the centre of a body (e.g. the centre of the Earth). See http://en.wikipedia.org/wiki/Standard_gravity for more information.

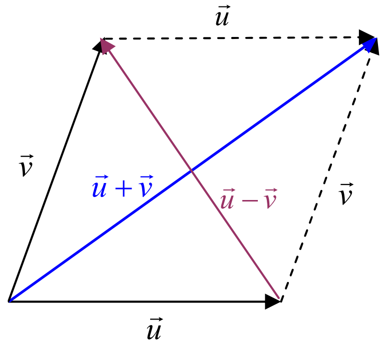
Newton's Third Law (Law of Action and Reaction)

For every ***action***, there is an ***equal but opposite reaction***. That is, matter interacts with matter. For each force exerted on one body, there is an equal but oppositely directed force on some other body interacting with it.

Review of Basic Properties of Geometric Vectors

Vector Addition and Subtraction

The vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are the diagonals of a parallelogram formed by the vectors $\vec{u}$ and $\vec{v}$.



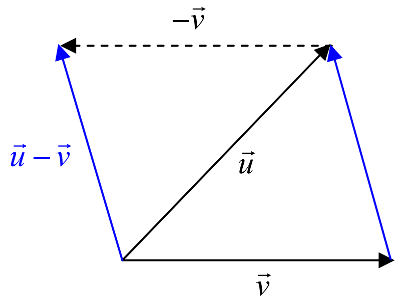
Triangle Inequalities

$$|\vec{u}+\vec{v}| \le |\vec{u}| + |\vec{v}|$$

$$|\vec{u}-\vec{v}| \le |\vec{u}| + |\vec{v}|$$

$$|\vec{u}+\vec{v}| \ge \big||\vec{u}| - |\vec{v}|\big|$$

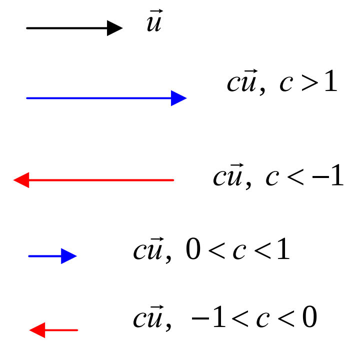
Vector Subtraction



$$\vec{u}-\vec{v} = \vec{u} + (-\vec{v})$$

$$\vec{v} + (\vec{u}-\vec{v}) = \vec{u}$$

Multiplying a Vector by a Scalar



$$|c\vec{u}| = |c||\vec{u}|$$

A ***geometric vector*** is a mathematical representation of any quantity that has both **magnitude** and **direction**.

Applications of Geometric Vectors

Forces

- The **vector sum** of all forces acting on a body is called the ***resultant*** or ***net force***.
- With respect to a given frame of reference, if a body is ***stationary*** or it is ***moving uniformly*** (i.e. with a constant velocity), the vector sum of all forces acting on the body ***must be the zero vector***.
- When using vectors to solve problems involving forces, it is useful to understand **Newton's Laws of Motion**.

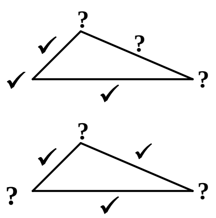
Relative Velocity

- As measured in a frame of reference attached to object B, the ***velocity of object A relative to (with respect to) object B*** is $\vec{v}_{AB} = \vec{v}_{AF} - \vec{v}_{BF}$, where $\vec{v}_{AF}$ and $\vec{v}_{BF}$ are the velocities of A and B respectively as measured in some other frame of reference F
- Keep in mind that measurements are relative to the frame of reference of an observer. Different observers in different frames of reference may not agree on their measurements of the same quantity. This is true of position, displacement, distance (length), velocity, mass, time and many other quantities.
- If we apply the above to objects in motion in the Earth's atmosphere (e.g. aircraft), then $\vec{v}_{OA} = \vec{v}_{OG} - \vec{v}_{AG}$ or $\vec{v}_{OG} = \vec{v}_{OA} + \vec{v}_{AG}$. Here $\vec{v}_{OG}$ represents velocity of the object relative to the ground, $\vec{v}_{OA}$ represents the velocity of the object relative to the air (atmosphere) and $\vec{v}_{AG}$ represents the velocity of the air (i.e. the wind velocity) with respect to the ground.

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos C$$

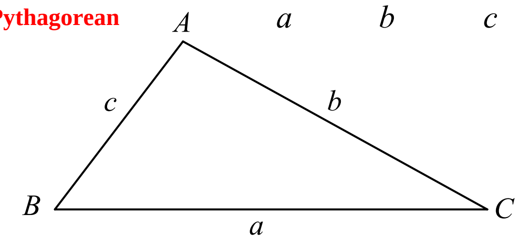
The law of cosines is a generalization of the Pythagorean Theorem.

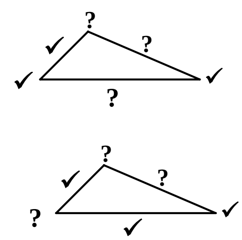


Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Beware of the ambiguous case.





Homework Exercises for Chapter 4 (Geometric Vectors)

We have reached the halfway point of the third unit. Now it is time to put into practice what we have learned. You should now be ready to do the following:

1. Read all of chapter four once again. Pay attention to the examples in the textbook, especially those that are different from the examples found in these notes or those given in class.
2. Read these notes once again. Ensure that you understand all examples and that you have answered all questions.
3. Do the following homework sets. Whenever necessary consult your textbook, these notes or any other resources.

Homework Set 1	*Homework Set 2*	*Homework Set 3*	*Homework Set 4*
p.127 #1, 2, 4, 5, 6	**p.128** #7, 8, 9, 10, 11	**p.128** #12, 13, 14	**p.144** #23, 24, 25
p.133 #1, 2, 3, 4, 5, 6	**p.133** #7, 8, 9, 11, 13, 15, 16	**p.133** #17, 19, 20, 21	
p.141 #1, 2, 3	**p.142** #5, 6a, 7a, 8, 9, 10, 12, 13	**p.143** #15, 16, 17, 20, 21, 22	**p.150** #11, 12, 13, 14
p.149 #1, 2, 3	**p.149** #3, 4, 5, 6	**p.150** #7, 8, 9, 10	

ALGEBRAIC VECTORS

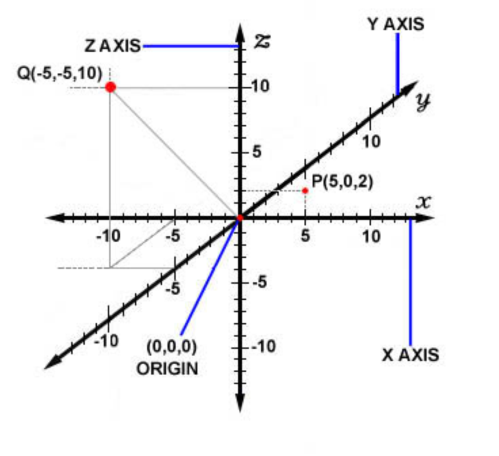
Introduction

As you should recall from unit 1, Descartes bridged the gap between algebra and geometry with his system of coordinates. This mathematical breakthrough also allows us to take geometric vectors into the realm of algebra. Since you are already all-too-familiar with a two-dimensional Cartesian (rectangular) coordinate system, we shall begin our discussion with its three-dimensional analogue.

Three-Dimensional Cartesian Coordinate System

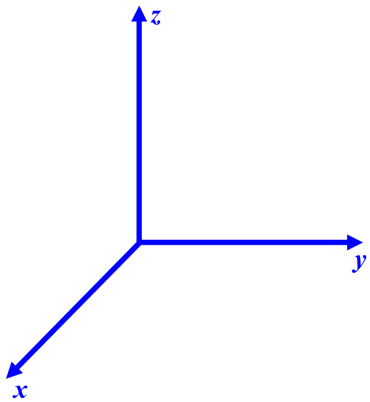
To extend the reach of the two-dimensional rectangular coordinate system, a third axis (the z-axis) is added. The z-axis passes through the origin and is perpendicular to the plane formed by the x-axis and the y-axis. In the diagram shown at the right, the x and y axes are shown as they would appear to us in the two-dimensional case, with the positive x-axis pointing to the right and the positive y-axis pointing "up."

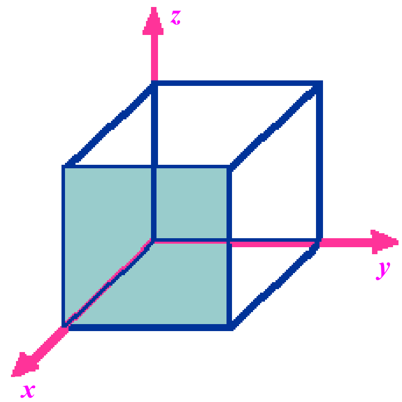
However, to give the appearance that the positive axes are facing us, we rotate the axes (in the diagram shown at the right) clockwise about the z-axis to obtain the more familiar view shown below (see next page).

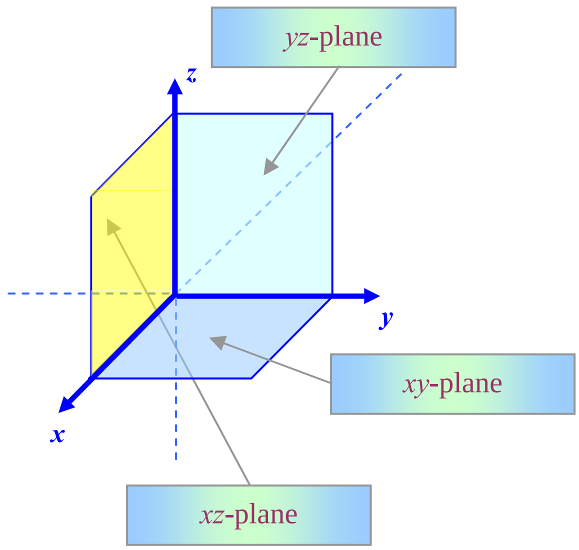


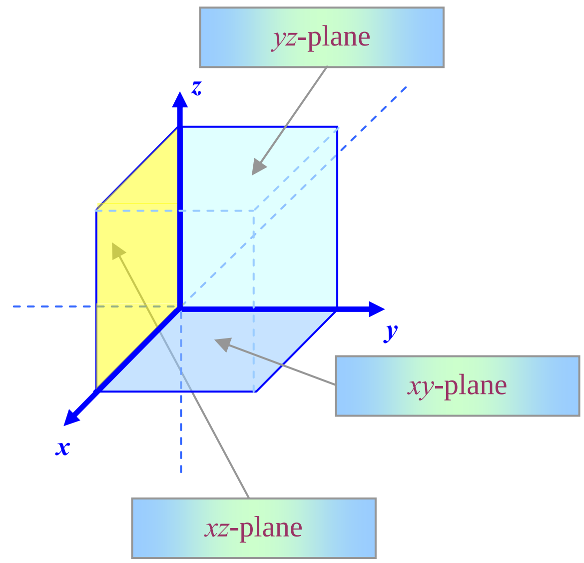
3 DIMENSIONAL CARTESIAN COORDINATE SYSTEM

The following diagrams show how we usually sketch the co-ordinate axes:



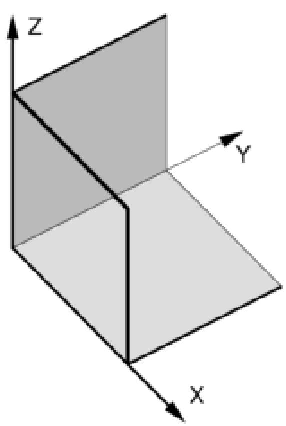






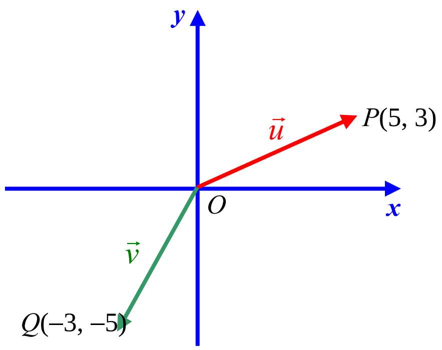
Note that for the purposes of these diagrams, the planes are shown as rectangles. You must keep in mind, however, that a ***plane*** is a ***flat surface*** that extends infinitely in all directions.

Here is another view that may help you visualize the *xy*-plane, the *xz*-plane and the *yz*-plane (collectively called the ***co-ordinate planes***).



Position Vectors

Up to this point, we have treated all vectors as ***free vectors***. As you should recall, a free vector's position in space is unimportant because it is completely determined by its magnitude and direction. Now we shall introduce the concept of a ***position vector***, a vector that always has its tail at the origin. Position vectors are useful because they provide us with a convenient method of locating points. Conversely, we can use the coordinates of a point to locate a position vector! A simple example will help to clarify these ideas.



The vectors $\vec{u}$ and $\vec{v}$ are ***free geometric vectors***. They can be placed anywhere in space as long as their magnitudes and directions remain unchanged. If we place the tails of $\vec{u}$ and $\vec{v}$ at the origin of a Cartesian coordinate system, then they become ***position vectors***.

- The ***position vector*** $\vec{u} = \overrightarrow{OP}$ locates the point $P(5, 3)$
- The ***position vector*** $\vec{v} = \overrightarrow{OQ}$ locates the point $Q(-3, -5)$

A great advantage of this idea is that the coordinates of a given point uniquely determine a position vector! For example, we can use the coordinates (5, 3) to uniquely identify the vector $\vec{u} = \overrightarrow{OP}$. This gives us an ***algebraic method*** of working with vectors. In addition, when a vector is expressed in algebraic form, it is not necessary to resolve it into mutually perpendicular components. The algebraic form of representing a vector ***includes*** the components of the vector!

Algebraic Vectors

Definitions

1. If $\overrightarrow{OP}$ is a position vector for the two-dimensional geometric vector $\vec{u}$ and point P has coordinates (a, b), then (a,b) is the ***algebraic representation*** of the ***two-dimensional geometric vector*** $\vec{u}$. We use the symbol $\mathbb{R}^2$ to denote the set of all two-dimensional vectors over the set of real numbers. The numbers a and b are called the ***components*** of the vector.

2. If $\overrightarrow{OP}$ is a position vector for the three-dimensional geometric vector $\vec{u}$ and point P has coordinates (a, b, c) then (a,b,c) is the ***algebraic representation*** of the ***three-dimensional geometric vector*** $\vec{u}$. We use the symbol $\mathbb{R}^3$ to denote the set of all three-dimensional vectors over the set of real numbers. The numbers a, b and c are called the ***components*** of the vector.

3. In general, we use the notation $(a_1, a_2, \ldots, a_n)$ to represent an ***n-dimensional algebraic vector***, where $n \in \mathbb{N}$. The symbol $\mathbb{R}^n$ is used to denote the set of all n-dimensional vectors over the set of real numbers. The numbers $a_1, a_2, \ldots, a_n$ are called the ***components*** of the vector.

Note

1. A set of vectors V is known as a ***vector space*** if the operations of ***vector addition*** and ***scalar multiplication*** are defined on V and if the operations have the following ten properties:

(a) If $\vec{u} \in V$ and $\vec{v} \in V$ then $\vec{u} + \vec{v} \in V$. (***Closure Under Vector Addition***)

(b) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (***Commutative Property of Vector Addition***)

(c) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (***Associative Property of Vector Addition***)

(d) There is an ***additive identity element*** $\vec{0}$ which has the property that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v} \in V$.

(e) For every vector $\vec{v} \in V$, there is an ***additive inverse element*** (i.e. "opposite" of $\vec{v}$) such that $\vec{v} + (-\vec{v}) = \vec{0}$.

(f) If $a \in \mathbb{R}$ and $\vec{v} \in V$, then $a\vec{v} \in V$ (***Closure Under Scalar Multiplication***)

(g) There is a ***scalar multiplicative identity element*** 1, which has the property that $1\vec{v} = \vec{v}$ for all $\vec{v} \in V$.

(h) $a(b\vec{v}) = (ab)\vec{v}$ (***Associative Property of Scalar Multiplication***)

(i) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ (***Distributive Property of Scalar Multiplication over Vector Addition***)

(j) $(a + b)\vec{v} = a\vec{v} + b\vec{v}$ (***Distributive Property of Scalar Multiplication over Scalar Addition***)

2. You may find it very difficult to visualize vectors of dimension greater than 3. Do not worry, this is quite normal! If you widen your perspective, however, and consider non-geometric applications of vectors, then it becomes easy to understand vectors of any dimension.

Consider an 8-dimensional algebraic vector such as (0, 1, 0, 0, 1, 1, 0, 1). This vector is not easy to interpret from a geometric standpoint but it is easy to understand in numerous other contexts. The following table lists just a few ways of understanding this vector and others like it. (If you have studied computer programming, you certainly should notice that ***algebraic vectors*** are analogous to ***one-dimensional arrays***!)

Various Interpretations of Multi-Dimensional Vectors

Application

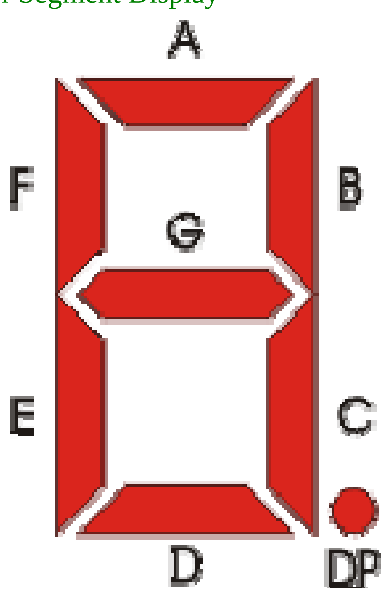
Code used to represent characters in computer systems.

Explanation

The vector (0, 1, 0, 0, 1, 1, 0, 1) is the binary code for the character "L"

Application

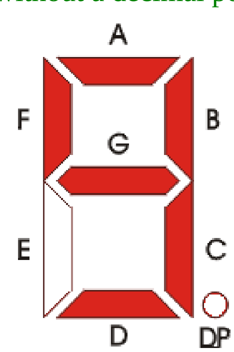
Seven-Segment Display



Explanation

A	B	C	D	E	F	G	DP
1	1	1	1	0	1	1	0

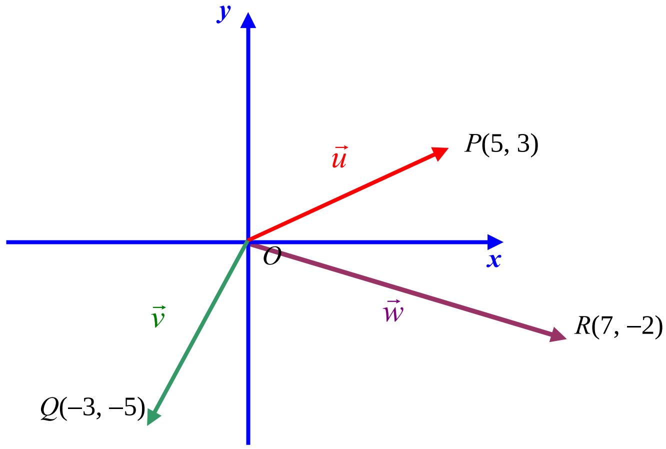
In this case, the vector (1, 1, 1, 1, 0, 1, 1, 0) means that all segments should be turned on except for "E" and "DP" ("1" means "on" and "0" means "off"). Thus, a "9" would be displayed, without a decimal point.



Calculating Magnitude and Direction of Vectors Expressed in Algebraic Form

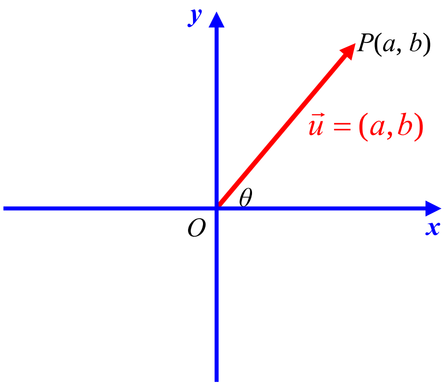
Instructions

1. Calculate the magnitude and direction of each of the vectors shown below. State the direction as an angle θ between 0° and 360° such that θ is measured counter-clockwise from the positive x-axis.

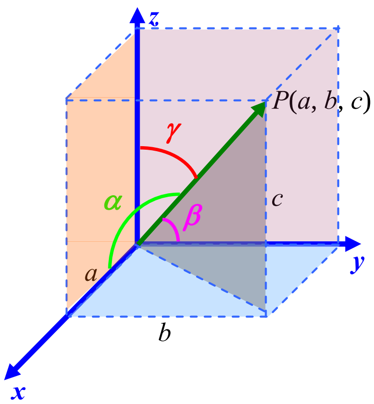


2. Now derive formulas for the magnitude and direction of $\vec{u}=(a,b)$. Express both $|\vec{u}|$ and θ in terms of a and b.

Hint: Use the Pythagorean Theorem to calculate $|\vec{u}|$ and trigonometry to calculate θ.



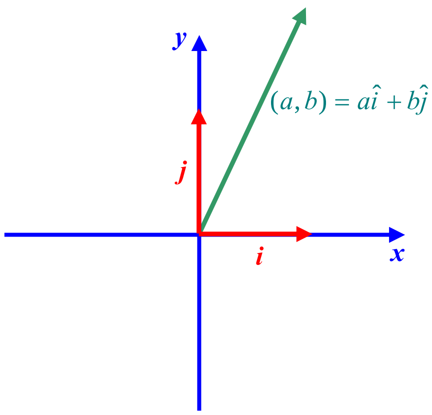
3. Now do the same for $\mathbb{R}^3$. Given the position vector $\vec{u}=\overrightarrow{OP}$ for the point $P(a, b, c)$, calculate $\left|\overrightarrow{OP}\right|$ in terms of a, b and c. In addition, calculate the ***direction angles*** α, β and γ in terms of a, b and c. (Note that α is the angle between the positive x-axis and the vector $\vec{u}$, β is the angle between the positive y-axis and the vector $\vec{u}$ and γ is the angle between the positive z-axis and the vector $\vec{u}$.) Can you find a relationship among the direction cosines? (The ***direction cosines*** are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.)



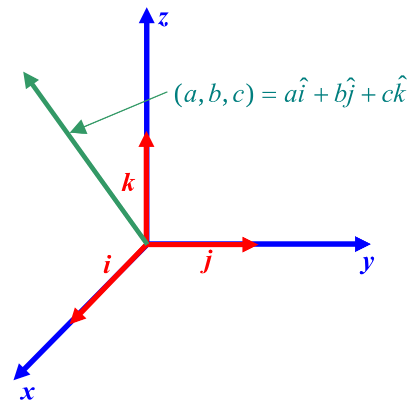
UNIT VECTORS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

Introduction

Unit vectors are important because it is so easy to work with them. A ***unit vector*** is any vector of magnitude 1. That is, any vector $\vec{u}$ with $|\vec{u}|=1$ is called a unit vector. In $\mathbb{R}^2$ and $\mathbb{R}^3$, it is especially convenient to work with unit vectors that lie along (i.e. are parallel to) the coordinate axes. These unit vectors are so important that we give them the special names ***i***, ***j*** and ***k*** ($\hat{i}$, $\hat{j}$ and $\hat{k}$ in our textbook). The diagrams below show the orientation of these unit vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.



- ***i*** (or $\hat{i}$) is a unit vector in the direction of the positive *x*-axis
- ***j*** (or $\hat{j}$) is a unit vector in the direction of the positive *y*-axis
- In $\mathbb{R}^3$, ***k*** (or $\hat{k}$) is a unit vector in the direction of the positive *z*-axis
- In $\mathbb{R}^2$, ***i*** = (1, 0) and ***j*** = (0, 1).
- In $\mathbb{R}^3$, ***i*** = (1, 0, 0), ***j*** = (0, 1, 0) and ***k*** = (0, 0, 1).



Ordered Pair (Triple) Notation or Unit Vector Notation?

It doesn't matter whether you use ordered pair notation or unit vector notation. They are equivalent in every respect! The choice depends only on convenience. If it is easier to use ordered pair notation, then do so. Otherwise, use unit vector notation.

More Examples

Read section 5.1 of our textbook to obtain more examples and more information.

OPERATIONS WITH VECTORS IN ALGEBRAIC FORM

Basis for a Vector Space

It doesn't take long to realize that any vector in $\mathbb{R}^2$ can be expressed in terms of the unit vectors ***i*** and ***j*** and that any vector in $\mathbb{R}^3$ can be expressed in terms of the unit vectors ***i***, ***j*** and ***k***. If we consider $\mathbb{R}^3$ for example, we can "travel" to any point *P*(*a*, *b*, *c*) by starting at the origin and moving only in the directions of the three coordinate axes. To arrive at the point *P*(*a*, *b*, *c*), we simply move *a* units in the *x*-direction, *b* units in the *y*-direction and *c* units in the *z*-direction. Using this method, we can reach any point in $\mathbb{R}^3$.

Any ***minimal*** set of vectors that can be used to express any vector in a vector space is called a ***basis*** for the vector space. (By ***minimal*** we mean that the set is as small as possible. For instance, any basis for $\mathbb{R}^3$ must have exactly 3 vectors. It is impossible to form a basis for $\mathbb{R}^3$ using fewer than 3 vectors. Moreover, any set of vectors that consists of 4 or more vectors is not considered a basis for $\mathbb{R}^3$ because it's larger than it needs to be.)

Notice that the number of vectors in a basis ***equals*** the dimension of the vector space. For example, $\mathbb{R}^2$ is two-dimensional because any basis for $\mathbb{R}^2$ has exactly two vectors.

Examples

1. Prove that two vectors are equal ***if and only if*** their respective Cartesian components are equal.

2. Prove that if $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$ and c is a scalar, then $\vec{a}+\vec{b}=(a_1+b_1,a_2+b_2,a_3+b_3)$ and $c\vec{a}=(ca_1,ca_2,ca_3)$. (**Hint:** Express each vector in as a sum of scalar multiples of ***i***, ***j*** and ***k***.)

3. Are the points $P(1, 2, 3)$, $Q(-2, 4, 6)$ and $R(0, 0, 3)$ collinear?

4. If the points $A(0, 3, 0)$ and $C(6, -1, 4)$ are opposite vertices of the parallelogram $ABCD$, and $B(5, 0, 0)$ is one of the other vertices, find the coordinates of the point D. In addition, describe the orientation of the parallelogram relative to the coordinate axes.

Additional Vector Operations – The Dot Product and the Cross Product

Introduction

In this section we shall learn about two additional operations, the ***dot product*** and the ***cross product***. The dot product is defined on $\mathbb{R}^n$ for all $n \in \mathbb{N}$. The cross product, however, is defined only on $\mathbb{R}^3$.

The dot product, also known as the ***scalar product*** or ***inner product***, is an extremely useful tool with many applications. One of its principle applications is to allow us to calculate the angle between two vectors without the use of complex geometry. If $\vec{u}$ and $\vec{v}$ are vectors, then the dot product of $\vec{u}$ and $\vec{v}$ is written $\vec{u}\cdot\vec{v}$. The dot product of two vectors ***always produces a scalar***, that is, $\vec{u}\cdot\vec{v}\in\mathbb{R}$.

The ***cross product*** of two vectors, on the other hand, ***always produces a vector***. It is defined in such a way that the result of finding the cross product of two vectors is a third vector that is perpendicular to each of the two original vectors. In other words, if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, then $\vec{u}\times\vec{v}$ (read "$\vec{u}$ cross $\vec{v}$") is a third vector in $\mathbb{R}^3$ such that $(\vec{u}\times\vec{v})\perp\vec{u}$ and $(\vec{u}\times\vec{v})\perp\vec{v}$.

Defining the Dot Product

As mentioned in a previous unit, mathematical definitions often seem arbitrary and senseless. Such is certainly the case with the dot product. Therefore, before we introduce its definition, we need to understand what motivated mathematicians to define $\vec{u}\cdot\vec{v}$ in such a seemingly strange way!

Before proceeding, we need to remember that like any other operation, the dot product is nothing more than a convenient tool for solving problems. In addition, if it is to be a useful tool, the dot product should have certain desirable properties. First of all, the dot product is "designed" in such a way that it has certain convenient properties as listed below:

Desired Properties of the Dot Product

1. If $\vec{u}\perp\vec{v}$, then $\vec{u}\cdot\vec{v}=0$.

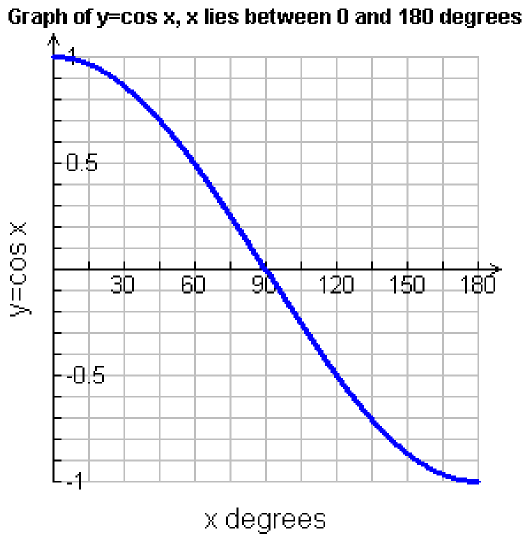
2. If $\vec{u}\parallel\vec{v}$, $|\vec{u}\cdot\vec{v}|=|\vec{u}||\vec{v}|$.

3. If $\vec{u}$ and $\vec{v}$ are expressed in algebraic form (i.e. in Cartesian or component form), there should be an easy way to calculate $\vec{u}\cdot\vec{v}$ ***without*** having to calculate the angle between $\vec{u}$ and $\vec{v}$.

Now let's examine the table below to gain a great deal of insight into how the dot product should be defined.

Angle between Vectors $\vec{u}$ and $\vec{v}$	*Value of $\vec{u}\cdot\vec{v}$*
0°	$\lvert\vec{u}\rvert\lvert\vec{v}\rvert = \lvert\vec{u}\rvert\lvert\vec{v}\rvert(1)$
90°	$0 = \lvert\vec{u}\rvert\lvert\vec{v}\rvert(0)$
180°	$-\lvert\vec{u}\rvert\lvert\vec{v}\rvert = \lvert\vec{u}\rvert\lvert\vec{v}\rvert(-1)$
θ	**?**

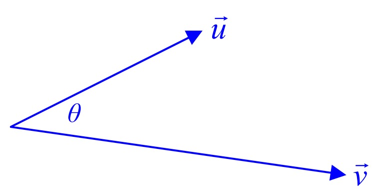
By observing the table carefully, we see that the dot product behaves much like the cosine function!



Therefore, it makes a great deal of sense to define the dot product in terms of $\cos\theta$.

Definition of the Dot Product

If $\vec{u}$ and $\vec{v}$ are vectors and θ is the non-reflex angle between the vectors ($0^\circ \le \theta \le 180^\circ$), then the dot product of $\vec{u}$ and $\vec{v}$, written $\vec{u}\cdot\vec{v}$, is defined as $\vec{u}\cdot\vec{v} = \lvert\vec{u}\rvert\lvert\vec{v}\rvert\cos\theta$.



Properties of the Dot Product

1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$ (Commutative Property of the Dot Product)
2. $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ (Distributive Property of the Dot Product over Vector Addition)
3. $a(\vec{u}\cdot\vec{v})=(a\vec{u})\cdot\vec{v}=\vec{u}\cdot(a\vec{v})$ (Associative Property)
4. $(a\vec{u})\cdot(b\vec{v})=ab(\vec{u}\cdot\vec{v})$ (Generalized Associative Property)
5. $\vec{u}\cdot\vec{u}=|\vec{u}|^2$
6. If $\vec{u}\neq\vec{0}$ and $\vec{v}\neq\vec{0}$, then $\vec{u}\perp\vec{v}$, if and only if $\vec{u}\cdot\vec{v}=0$.
7. $\vec{u}\parallel\vec{v}$ if and only if $|\vec{u}\cdot\vec{v}|=|\vec{u}||\vec{v}|$ (i.e. $\vec{u}\parallel\vec{v}$ if and only if $\vec{u}\cdot\vec{v}=\pm|\vec{u}||\vec{v}|$)
8. $\hat{i}\cdot\hat{i}=1$, $\hat{j}\cdot\hat{j}=1$, $\hat{k}\cdot\hat{k}=1$
9. $\hat{i}\cdot\hat{j}=0$, $\hat{i}\cdot\hat{k}=0$, $\hat{j}\cdot\hat{k}=0$
10. If $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ then $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3$.

Proof

The proofs of properties 1 through 9 are left up to you. The proof of property 10 is given below. Note that the proof of property 10 relies on most of properties 1 through 9.

$$\begin{aligned}\vec{u}\cdot\vec{v}&=(u_1,u_2,u_3)\cdot(v_1,v_2,v_3)\\&=(u_1\hat{i}+u_2\hat{j}+u_3\hat{k})\cdot(v_1\hat{i}+v_2\hat{j}+v_3\hat{k})\\&=u_1v_1(\hat{i}\cdot\hat{i})+u_1v_2(\hat{i}\cdot\hat{j})+u_1v_3(\hat{i}\cdot\hat{k})+u_2v_1(\hat{j}\cdot\hat{i})+u_2v_2(\hat{j}\cdot\hat{j})+u_2v_3(\hat{j}\cdot\hat{k})+u_3v_1(\hat{k}\cdot\hat{i})+u_3v_2(\hat{k}\cdot\hat{j})+u_3v_3(\hat{k}\cdot\hat{k})\\&=u_1v_1(1)+u_1v_2(0)+u_1v_3(0)+u_2v_1(0)+u_2v_2(1)+u_2v_3(0)+u_3v_1(0)+u_3v_2(0)+u_3v_3(1)\\&=u_1v_1+u_2v_2+u_3v_3\end{aligned}$$

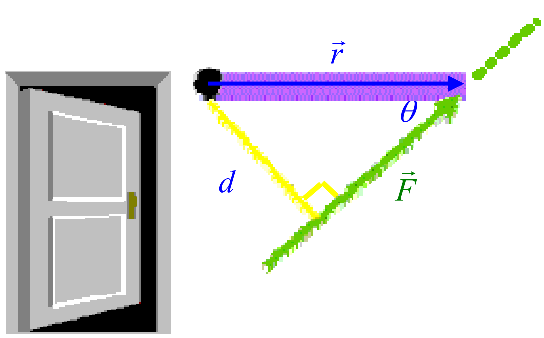
Examples

Carefully study examples 1 to 4 on pages 175-177 of our textbook.

Defining the Cross Product in such a way that it Models Torque

Unlike the other two products on vector spaces that we have studied, the cross product is defined only on $\mathbb{R}^3$. It is motivated by the physical phenomenon of rotational motion. Whenever a force causes linear motion (motion along a straight line), we can calculate the displacement (change in position) of an body simply by knowing the magnitude and the direction of the force.

In ***rotational motion***, however, a sufficiently strong force ***will cause a change in rotational motion***, not a change in position. In addition to knowing the magnitude and direction of the force, it is also necessary to know ***the point at which the force is applied***. The following examples should help you understand why.



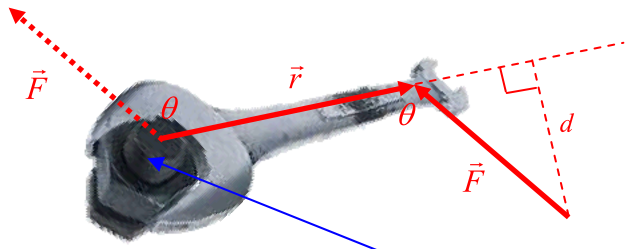
Consider a door that is closed. To open the door, a change in its rotational motion must take place. What will cause the largest possible change in its rotational motion?

- as large a force as possible
- in the correct direction
- and ***applied at the correct point***

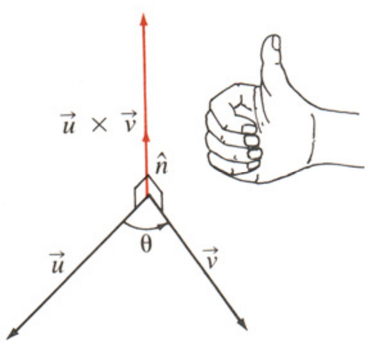
Consider a "top view" of the door

$\vec{F}$ $\rightarrow$ force vector, θ $\rightarrow$ angle between force and displacement vectors
$\vec{r}$ $\rightarrow$ displacement vector from axis of rotation to point of application of the force
d $\rightarrow$ the perpendicular distance from the center of rotation to the line of action of the force

Using the Idea of Torque to decide what the Direction of $\vec{u}\times\vec{v}$ should be



Both the nut and bolt are ***right-threaded***. This means that the bolt or nut must be turned ***clockwise*** in order to be tightened.



This diagram shows the orientation of $\vec{u}\times\vec{v}$ given $\vec{u}$ and $\vec{v}$. Notice that the right hand rule is being applied. The fingers of the right hand "curl" from $\vec{u}$ toward $\vec{v}$. The thumb points in the direction of $\vec{u}\times\vec{v}$. Note that $\vec{v}\times\vec{u}$, points in the opposite direction of $\vec{u}\times\vec{v}$.

Consider a wrench tightening a right-threaded nut or bolt. The ***rotational motion of the nut or bolt*** will depend on the magnitude and the direction of the applied force as well as the ***point*** at which the force is applied. In other words, the "strength" of the "turning effect" of the wrench is determined by how strong the force is, its direction and where it is applied. We call this "turning effect" of the applied force the ***torque*** or the ***moment of the force*** about the ***axis of rotation***.

Torque is a vector quantity that is perpendicular to both $\vec{F}$ and $\vec{r}$. Its direction is determined by the "right hand rule" for right-handed coordinate systems. In the diagram to the left, the torque is directed into the page at right angles to both $\vec{F}$ and $\vec{r}$. The direction of the torque is the same as the direction that the nut or bolt would move when being tightened.

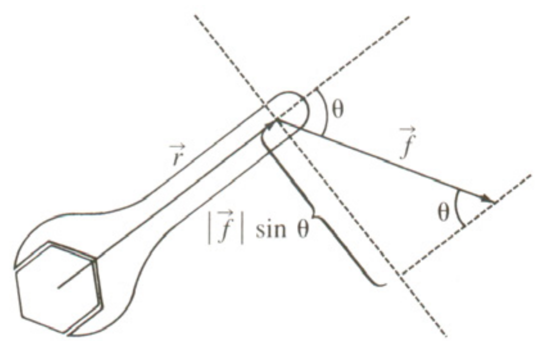
In general, the ***torque*** $\vec{\tau}$ (or the ***moment of force***) about a given axis of rotation is defined as

$$\vec{\tau}=\vec{r}\times\vec{F}.$$

In this equation, $\vec{r}$ represents the displacement from the axis of rotation to the point of application of the force and $\vec{F}$ represents the applied force. Note that sometimes the symbols $\vec{T}$ and $\vec{M}$ are used to represent torque instead of $\vec{\tau}$.

Using the Idea of Torque to decide what the Magnitude of $\vec{u}\times\vec{v}$ should be

Suppose that a wrench is tightening a right-threaded bolt, as shown in the diagram at the right. As discussed above, the ***torque*** $\vec{\tau}$ produced by the force $\vec{f}$ is a vector that is directed "into" the page, the same direction that a right-threaded bolt would move when being tightened. (Torque can be thought of as a measure of the ***effectiveness*** of the force in producing a rotation about the axis.)



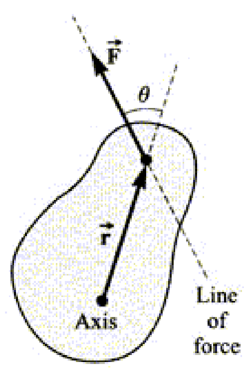
What would we expect the magnitude of this vector to be? Since the amount of torque depends on

(1) the distance between the bolt and the point at which the force is applied (i.e. $|\vec{r}|$)

(2) the strength of the component of $\vec{f}$ directed perpendicular* to the wrench (i.e. $|\vec{f}|\sin\theta$),

it is reasonable to expect the magnitude to be $|\vec{r}||\vec{f}|\sin\theta$

Therefore, it's sensible to expect $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$. We shall prove this shortly.



*Note that no contribution to the torque is made by the component of $\vec{f}$ that is parallel to $\vec{r}$.

Determining the Components of $\vec{u}\times\vec{v}$

Given $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$, how can we calculate $\vec{u}\times\vec{v}$? That is, how can we find a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$ and that accurately models the physical quantity of torque?

Let $\vec{a}=(x,y,z)$ represent such a vector. That is, let $\vec{a}=(x,y,z)$ represent any vector that is ***perpendicular to*** both $\vec{u}$ and $\vec{v}$. Then, $\vec{a}\cdot\vec{u}=0$ and $\vec{a}\cdot\vec{v}=0$. Therefore,

$$u_1x+u_2y+u_3z=0 \quad (1) \text{ and}$$
$$v_1x+v_2y+v_3z=0 \quad (2)$$

This is a ***system of 2 linear equations in 3 unknowns***. As we shall learn in the next unit, each of these equations represents a ***plane*** in $\mathbb{R}^3$. Unless the two planes are parallel, they will intersect in a line. The line of intersection of the two planes is the geometric representation of the solution of the system. Since there are an infinite number of points on a line, we should expect an infinite number of solutions to the system.

$v_3\times(1)-u_3\times(2)$, $\quad (u_1v_3-u_3v_1)x+(u_2v_3-u_3v_2)y=0 \quad (3)$

$v_1\times(1)-u_1\times(2)$, $\quad (u_2v_1-u_1v_2)y+(u_3v_1-u_1v_3)z=0 \quad (4)$

Rearranging (3), $\quad x=\dfrac{(u_2v_3-u_3v_2)y}{u_3v_1-u_1v_3} \quad (5)$

Rearranging (4), $\quad z=\dfrac{(u_1v_2-u_2v_1)y}{u_3v_1-u_1v_3} \quad (6)$

Now we have expressed both x and z in terms of y. Since we know that the system of equations has an infinite number of solutions, we can find ***one particular solution*** by choosing a value for y. If we choose this value carefully, we shall be able to find formulas that are as simple as possible.

By choosing $y=u_3v_1-u_1v_3$ and substituting into equations (5) and (6), we obtain $x=u_2v_3-u_3v_2$ and $z=u_1v_2-u_2v_1$. Finally, we are ready to state a method for computing the cross product.

Definition of the Cross Product (also known as the Gibbs Vector Product)

Given $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$, $\vec{u}\times\vec{v}=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$.

Example

Find a vector perpendicular to both $\vec{u}=(1,2,3)$ and $\vec{v}=(-5,2,-1)$.

Solution

The formula given above is a little difficult to remember so here is a simple method for remembering how to apply it. First write the components of the first vector directly above the second. Then follow the arrows, always following **blue** ***before*** **green**.

1 –5	2 2	3 –1	To calculate the x-component of $\vec{u}\times\vec{v}$, ignore the x-components of $\vec{u}$ and $\vec{v}$ and follow the arrows. 2(–1) – 3(2) = –8
1 –5	2 2	3 –1	To calculate the y-component of $\vec{u}\times\vec{v}$, ignore the y-components of $\vec{u}$ and $\vec{v}$ and follow the arrows. 3(–5) – 1(–1) = –14
1 –5	2 2	3 –1	To calculate the z-component of $\vec{u}\times\vec{v}$, ignore the z-components of $\vec{u}$ and $\vec{v}$ and follow the arrows. 1(2) – 2(–5) = 12

Therefore, $\vec{u}\times\vec{v}=(-8,-14,12)$, which is perpendicular to both $\vec{u}$ and $\vec{v}$.

Note

To calculate the y-component of $\vec{u}\times\vec{v}$, you may also perform the operations in the same order as for the x and y components, but you must remember to change the sign of the value that you obtain.

Does our Definition of the Cross Product Accurately Model Torque?

Before we accept blindly our definition of $\vec{u}\times\vec{v}$, we should confirm that it behaves as expected. Thus, we should verify that the direction of $\vec{u}\times\vec{v}$ corresponds with our notion of tightening a right-threaded bolt and that the magnitude of $\vec{u}\times\vec{v}$ is indeed equal to $|\vec{u}||\vec{v}|\sin\theta$.

Calculating $|\vec{u}\times\vec{v}|$

First, let's check that $|\vec{u}\times\vec{v}|$ really does equal $|\vec{u}||\vec{v}|\sin\theta$. This is a somewhat tricky, tedious and messy calculation. Please refrain from yawning or snoring!

If $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$, $\vec{u}\times\vec{v}=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$. Therefore,

$$
\begin{aligned}
|\vec{u}\times\vec{v}|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\
&= u_2^2v_3^2-2u_2v_2u_3v_3+u_3^2v_2^2+u_3^2v_1^2-2u_1v_1u_3v_3+u_1^2v_3^2+u_1^2v_2^2-2u_1v_1u_2v_2+u_2^2v_1^2
\end{aligned}
$$

At this point, we scratch our heads and wonder where this is going. To avoid a great deal of frustration, let's try a method employed by the ancient Greeks whenever the search for a logical "path" to the desired conclusion seemed to lead to nothing but dead ends. **Let's begin with the conclusion and work our way back to the initial premise. Then we can try to *reverse* the steps.**

$$|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$$

Begin with the conclusion.

$$|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2\sin^2\theta$$

Square both sides to avoid having to use the square root symbol.

$$=|\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta)$$

Use a trig identity to change from "sin" to "cos." This may allow us to introduce the dot product, which can be expressed in terms of "cos." This ***may*** help us because the dot product has so many useful properties.

$$
\begin{aligned}
&=|\vec{u}|^2|\vec{v}|^2-|\vec{u}|^2|\vec{v}|^2\cos^2\theta\\
&=|\vec{u}|^2|\vec{v}|^2-(\vec{u}\cdot\vec{v})^2\\
&=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-((u_1,u_2,u_3)\cdot(v_1,v_2,v_3))^2\\
&=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2\\
&=u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2\\
&\qquad -u_1^2v_1^2-u_1v_1u_2v_2-u_1v_1u_3v_3-u_2v_2u_1v_1-u_2^2v_2^2-u_2v_2u_3v_3-u_3v_3u_1v_1-u_3v_3u_2v_2-u_3^2v_3^2\\
&=u_1^2v_2^2-2u_1v_1u_2v_2+u_2^2v_1^2+u_1^2v_3^2-2u_1v_1u_3v_3+u_3^2v_1^2+u_2^2v_3^2-2u_2v_2u_3v_3+u_3^2v_2^2\\
&\qquad +u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2-u_1^2v_1^2-u_2^2v_2^2-u_3^2v_3^2\\
&=u_2^2v_3^2-2u_2v_2u_3v_3+u_3^2v_2^2+u_1^2v_3^2-2u_1v_1u_3v_3+u_3^2v_1^2+u_1^2v_2^2-2u_1v_1u_2v_2+u_2^2v_1^2\\
&=(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2
\end{aligned}
$$

Lo and behold, we are back where we started! ***By reversing these steps, therefore, we can obtain the desired result***. (See next page.)

If $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$, $\vec{u}\times\vec{v}=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$. Therefore,

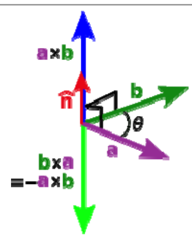
The trick that we discovered by working backwards is to add and subtract the terms shown here.

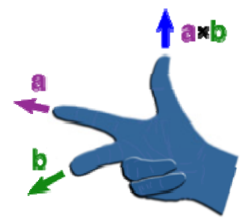
$$
\begin{aligned}
\left|\vec{u}\times\vec{v}\right|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\
&= {u_2}^2{v_3}^2-2u_2v_2u_3v_3+{u_3}^2{v_2}^2+{u_3}^2{v_1}^2-2u_1v_1u_3v_3+{u_1}^2{v_3}^2+{u_1}^2{v_2}^2-2u_1v_1u_2v_2+{u_2}^2{v_1}^2\\
&= {u_1}^2{v_2}^2-2u_1v_1u_2v_2+{u_2}^2{v_1}^2+{u_1}^2{v_3}^2-2u_1v_1u_3v_3+{u_3}^2{v_1}^2+{u_2}^2{v_3}^2-2u_2v_2u_3v_3+{u_3}^2{v_2}^2\\
&\qquad +u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2-u_1^2v_1^2-u_2^2v_2^2-u_3^2v_3^2\\
&= {u_1}^2{v_1}^2+{u_1}^2{v_2}^2+{u_1}^2{v_3}^2+{u_2}^2{v_1}^2+{u_2}^2{v_2}^2+{u_2}^2{v_3}^2+{u_3}^2{v_1}^2+{u_3}^2{v_2}^2+{u_3}^2{v_3}^2\\
&\qquad -{u_1}^2{v_1}^2-u_1v_1u_2v_2-u_1v_1u_3v_3-u_2v_2u_1v_1-{u_2}^2{v_2}^2-u_2v_2u_3v_3-u_3v_3u_1v_1-u_3v_3u_2v_2-{u_3}^2{v_3}^2\\
&= ({u_1}^2+{u_2}^2+{u_3}^2)({v_1}^2+{v_2}^2+{v_3}^2)-(u_1v_1+u_2v_2+u_3v_3)^2\\
&= \left|\vec{u}\right|^2\left|\vec{v}\right|^2-(\vec{u}\cdot\vec{v})^2\\
&= \left|\vec{u}\right|^2\left|\vec{v}\right|^2-\left|\vec{u}\right|^2\left|\vec{v}\right|^2\cos^2\theta\\
&= \left|\vec{u}\right|^2\left|\vec{v}\right|^2(1-\cos^2\theta)\\
&= \left|\vec{u}\right|^2\left|\vec{v}\right|^2\sin^2\theta
\end{aligned}
$$

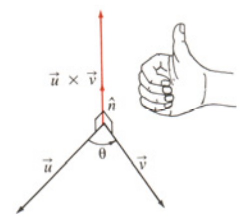
By taking the square root of both sides, we finally obtain the result, $\left|\vec{u}\times\vec{v}\right|=\left|\vec{u}\right|\left|\vec{v}\right|\sin\theta$

Summary

Given $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)$, $\vec{a}\times\vec{b}=(a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1)$.







Properties of the Cross Product

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$, then

1. $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$ (Anti-commutative Law)
2. $\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times\vec{v}+\vec{u}\times\vec{w}$ (Distributive Law of the Cross Product over Vector Addition)
3. $k(\vec{u}\times\vec{v})=(k\vec{u})\times\vec{v}=\vec{u}\times(k\vec{v})$
4. $\vec{u}\times\vec{u}=\vec{0}$
5. $\vec{u}\parallel\vec{v}$ if and only if $\vec{u}\times\vec{v}=\vec{0}$ (This is not the most efficient method for determining whether vectors are parallel!)
6. $\hat{i}\times\hat{j}=\hat{k}$, $\hat{i}\times\hat{k}=-\hat{j}$, $\hat{j}\times\hat{k}=\hat{i}$
7. $\vec{u}\cdot(\vec{v}\times\vec{w})=(\vec{u}\times\vec{v})\cdot\vec{w}$ (Products of this form are called "***triple scalar products***")

Questions

1. Prove properties 1 to 7 listed above.
2. We have already verified that $\left|\vec{u}\times\vec{v}\right|=\left|\vec{u}\right|\left|\vec{v}\right|\sin\theta$. Now use examples to show that $\vec{u}\times\vec{v}$ points in the right direction.
3. Is the cross product associative?
4. What unit could be used to measure torque? Is this unit used to measure any other physical quantities that you have studied?
5. You may have heard about torque in automotive advertising. In addition, you may have heard about horsepower. Do some research to find out how these two quantities are used to measure the "power" of a car.
6. Use a geometric argument to explain why $\vec{u}\cdot(\vec{u}\times\vec{v})=0$ and $\vec{v}\cdot(\vec{u}\times\vec{v})=0$

An Alternative Method of Remembering how to find the Cross Product of Two Vectors

We can use the concept of the ***determinant of a matrix*** as a handy method for remembering how to calculate the cross product of two vectors. (We shall study matrices in the next unit.)

Let $\vec{a}=(a_1,a_2,a_3)=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=(b_1,b_2,b_3)=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Then,

$$\vec{a}\times\vec{b}=\det\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}=\det\begin{pmatrix}a_2 & a_3\\ b_2 & b_3\end{pmatrix}\hat{i}-\det\begin{pmatrix}a_1 & a_3\\ b_1 & b_3\end{pmatrix}\hat{j}+\det\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}\hat{k}\text{, where }\det\begin{pmatrix}x & y\\ z & w\end{pmatrix}=xw-yz\text{.}$$

Example

$\vec{a}=2\hat{i}-5\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+3\hat{j}-2\hat{k}$

$$\vec{a}\times\vec{b}=\det\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -5 & 3\\ -1 & 3 & -2\end{pmatrix}=\det\begin{pmatrix}-5 & 3\\ 3 & -2\end{pmatrix}\hat{i}-\det\begin{pmatrix}2 & 3\\ -1 & -2\end{pmatrix}\hat{j}+\det\begin{pmatrix}2 & -5\\ -1 & 3\end{pmatrix}\hat{k}$$

$$=[(-5)(-2)-3(3)]\hat{i}-[2(-2)-3(-1)]\hat{j}+[2(3)-(-5)(-1)]\hat{k}$$

$$=\hat{i}+\hat{j}+\hat{k}$$

Applications of the Dot Product and Cross Product

Introduction

In this section we shall discuss applications of the dot product and cross product including, area, volume, projections, work and torque. Since we discussed torque extensively in the previous section, we shall focus on the others in this section.

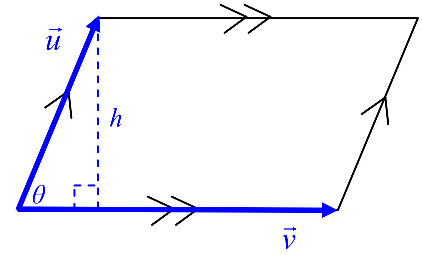
Area of a Parallelogram

By observing the diagram at the right, one readily notices that $h=|\vec{u}|\sin\theta$. In addition, the length of the base is equal to $|\vec{v}|$. Therefore the area A of the parallelogram is given by

$$A=bh=|\vec{v}|(|\vec{u}|\sin\theta)=|\vec{u}||\vec{v}|\sin\theta=|\vec{u}\times\vec{v}|$$

Therefore, the area of a parallelogram having sides $\vec{u}$ and $\vec{v}$ is given by

$$A=|\vec{u}\times\vec{v}|$$



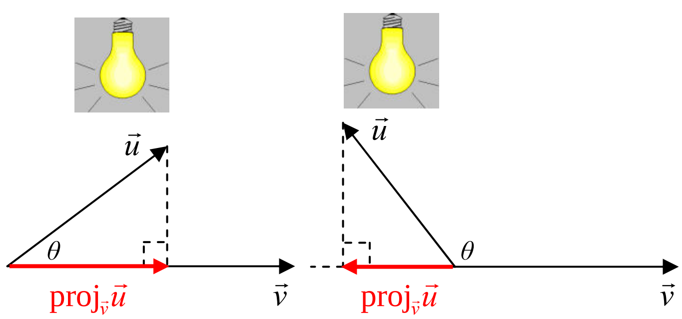
Projections

Imagine a light source directly above a vector $\vec{u}$ and a vector $\vec{v}$ perpendicular to the light rays emitted by the source. The vector $\vec{u}$ would cast a "shadow" on $\vec{v}$ known as the ***projection of $\vec{u}$ on $\vec{v}$***. (You can also visualize projections by dropping a perpendicular from each point on $\vec{u}$ onto $\vec{v}$.)

$$|\text{proj}_{\vec{v}}\vec{u}|=\left||\vec{u}|\cos\theta\right|$$

$$=\frac{\left||\vec{u}|\cos\theta\right||\vec{v}|}{|\vec{v}|}=\frac{\left||\vec{u}||\vec{v}|\cos\theta\right|}{|\vec{v}|}=\frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}$$

$$\therefore\ \text{proj}_{\vec{v}}\vec{u}=\frac{(\vec{u}\cdot\vec{v})}{|\vec{v}|}\hat{v}=\frac{(\vec{u}\cdot\vec{v})}{|\vec{v}|}\left(\frac{\vec{v}}{|\vec{v}|}\right)=\frac{(\vec{u}\cdot\vec{v})\vec{v}}{|\vec{v}|^2}$$



$$\therefore\ \text{proj}_{\vec{v}}\vec{u}=\frac{(\vec{u}\cdot\vec{v})\vec{v}}{|\vec{v}|^2}$$

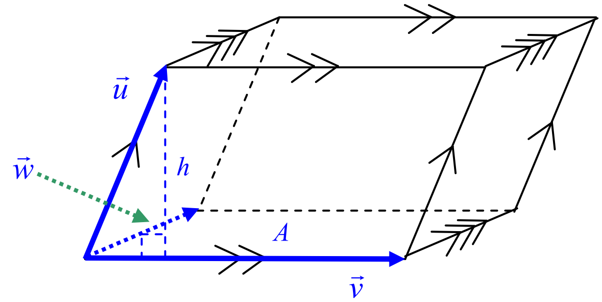
Volume of a Parallelepiped

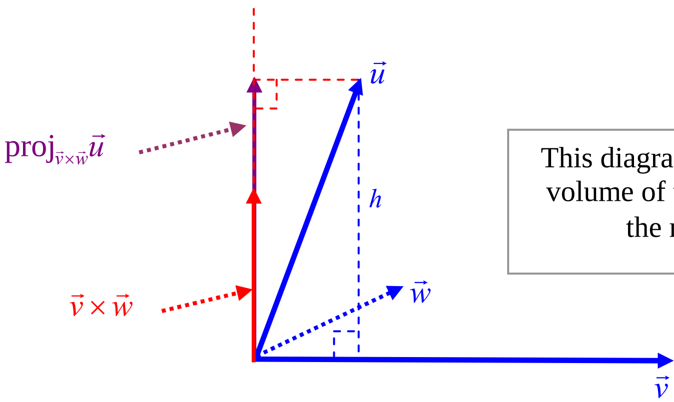
A parallelepiped is a solid, the opposite faces of which are parallel and congruent parallelograms. In the diagram at the right, the edges of the parallelepiped are the non-coplanar vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$. Let A represent the area of the base of the parallelepiped and let V represent the volume of the parallelepiped.

Then, since a parallelepiped is a ***regular solid***,

$$\begin{aligned} V &= Ah \\ &= |\vec{v}\times\vec{w}|\,|proj_{\vec{v}\times\vec{w}}\vec{u}| \\ &= |\vec{v}\times\vec{w}|\frac{|\vec{u}\cdot(\vec{v}\times\vec{w})|}{|\vec{v}\times\vec{w}|} \\ &= |\vec{u}\cdot(\vec{v}\times\vec{w})| \end{aligned}$$

(***absolute value of triple scalar product***)





This diagram shows only the vectors involved in calculating the volume of the parallelepiped. Notice that the height is equal to the magnitude of the projection of $\vec{u}$ onto $\vec{v}\times\vec{w}$.

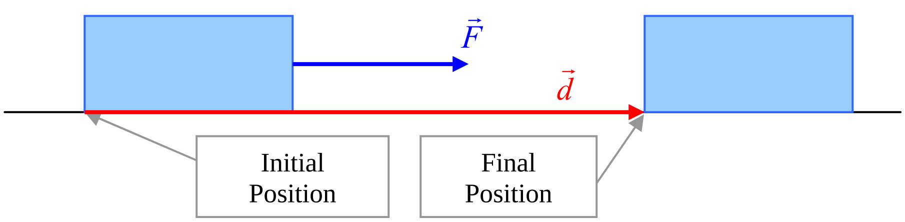
Work

Scientific Meaning of Work (for Motion along a Straight Line)

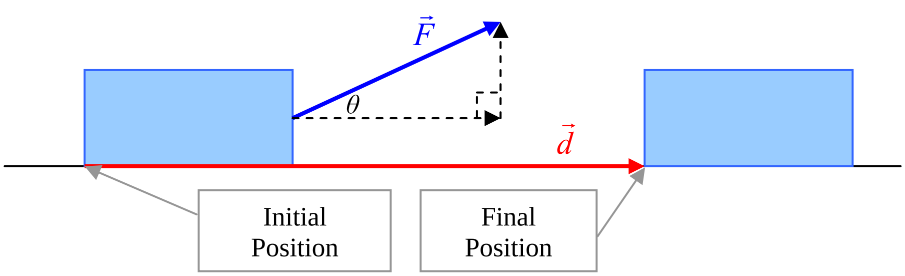
From a scientific perspective, ***work*** is done whenever a **force** causes a **displacement**.

In a scientific context, the word "work" has a much narrower meaning than in everyday life. If you happened to be standing still while holding up a heavy body, you certainly would feel as if you were doing hard work! However, a physicist would argue that you are doing no work because the body is not displaced (moved).

If you have studied physics, you have probably learned that work is the product of force and distance ($W = Fd$, where F represents the net force acting on the body). This formula works well as long as the body is displaced in the ***same direction*** as that of the net force. What happens if the force is applied in a different direction? In this case, we need to take into account the angle between the force and displacement vectors. (See diagrams below.)



When the direction of $\vec{F}$ is the same as that of $\vec{d}$, the work done by $\vec{F}$ is equal to $W = |\vec{F}||\vec{d}|$.



When the directions of $\vec{F}$ and $\vec{d}$ are not the same, the angle between the vectors must be taken into account. In this case, only the horizontal component of $\vec{F}$ contributes to the work done. The vertical component does no work because the object moves in the horizontal direction only. In this case, work is equal to

$$W = (|\vec{F}|\cos\theta)|\vec{d}| = |\vec{F}||\vec{d}|\cos\theta = \vec{F}\cdot\vec{d}$$

Scientific Definition of Work (for Motion along a Straight Line)

Suppose that a force $\vec{F}$ is applied to a body at a given angle θ, causing a displacement $\vec{d}$. If the displacement is due to motion along a straight line, then the work W done on the body by the force $\vec{F}$ is defined as

$$W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}|\cos\theta .$$

(If the motion follows a path other than a straight line or if the force changes over time, the above definition no longer holds. In this case, a general definition of work can be given in terms of a ***line integral***, a concept from advanced calculus.)

Unit of Work

Recall that $W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}|\cos\theta$. Since the basic unit for measuring $|\vec{F}|$ is the ***Newton*** and the basic unit for measuring $|\vec{d}|$ is the ***metre***, the basic unit of work is called the ***Newton-metre***. In keeping with the idea that unit names should be as simple as possible, a Newton-metre is more commonly known as a ***Joule***.

Intuitive Understanding of Work (for Motion along a Straight Line and v considerably smaller than light speed)

You may have noticed that ***work*** and ***energy*** are both measured in ***Joules***. The following points should help you to understand the close relationship between work and energy.

- Work is the ***amount of energy transferred to a body*** by a force.
- Suppose that two or more forces act on a body and that $\vec{F}_{net}$ represents the resultant force. Then the ***work done*** by $\vec{F}_{net}$ is the ***change in kinetic energy of the body***. That is,

$$\begin{aligned} W_{net} &= \Delta E_k \\ &= E_{kf} - E_{ki} \\ &= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 , \\ &= \frac{1}{2}m(v_f^2 - v_i^2) \end{aligned}$$

  where E_{ki} and E_{kf} represent the initial and final kinetic energies of the body respectively, and where v_i and v_f represent its initial and final velocities respectively.
- If a ***body moves uniformly*** (in a straight line, at a constant speed) the ***net work done is zero*** (since the net force is $\vec{0}$).
- Thus, we can also view the ***net work done*** as the ***energy required to accelerate a body*** from one position to another.

  For instance, if a car ***accelerates*** (which means there is a net force acting on the car) through a distance of 100 m along a straight line, a certain amount of work W_{net} is done. The quantity W_{net} represents the energy required to accelerate the car through a distance of 100 m assuming 100% energy efficiency. In reality, the energy expenditure is far greater than W_{net} because a great deal of energy is dissipated as heat.

  Specifically, the energy that is used to make a car move comes from the combustion of gasoline. A great deal of energy is released during this combustion process but only a relatively small portion of it is converted into the kinetic energy of the moving vehicle. Some of it is transformed into other forms of energy, which collectively are known as ***heat*** (energy that does no work). Mechanical engineers constantly grapple with the problem of using more of the energy of combustion to move a car and losing less of the energy to heat. That is, their goal is to transform as much of the energy of combustion as possible into kinetic energy.

See http://en.wikipedia.org/wiki/Mechanical_work for a detailed explanation of work.

Torque Revisited

Since we have already discussed the concept of torque extensively, at this point we shall investigate an example of torque in the automotive world.

Example

Shown below are some specifications for a Ferrari 360 Modena

Year	2000
Model	360 Modena
Engine	3.6 L (220 cubic inches) DOHC V-8
Weight	3065 lbs (1393 kg)
Horsepower	395 bhp (296 kW) @ 8500 rpm
Torque	275 lb-ft (373 J) @ 4750 rpm
0 - 60 mph (0 - 96.5 km/h)	3.9 s
1/4 mile (0.40225 km)	12.2 s

The following is a great site for performing conversions between imperial and metric units:
http://www.metrication.com

Power of the Ferrari 360 Modena

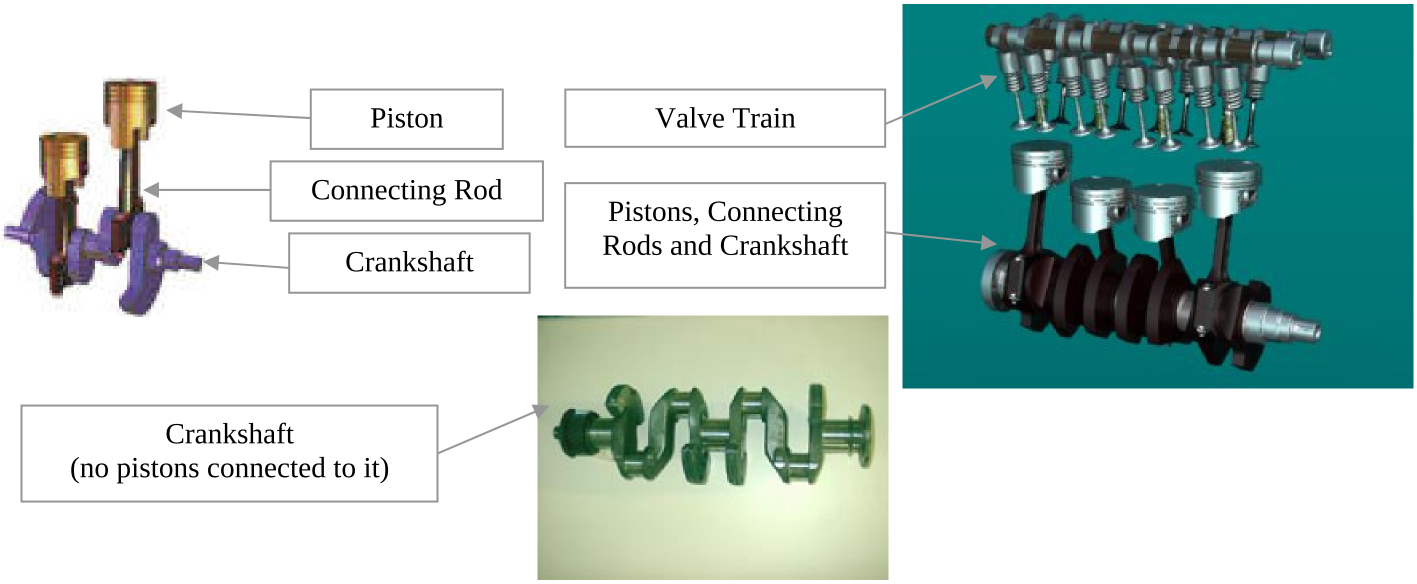
In science, power is defined as the ***rate of change of work done with respect to time***. In other words, you can think of power as the amount of work done per unit time. The Ferrari 360 Modena shown on the previous page has a maximum power output of 296 kW (kilowatts), which means that it can do up to 296 kJ (kilojoules) of work per second. (Note that 1 W = 1 J/s).

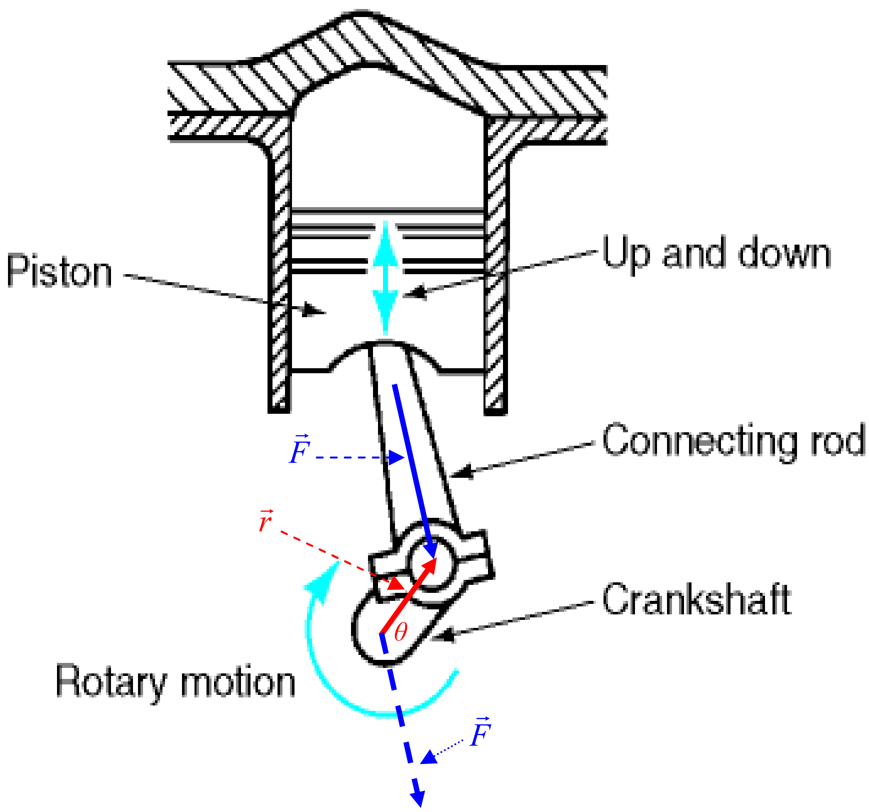
Torque of the Ferrari 360 Modena

As we know, ***torque is the "turning effect" produced by a force*** in rotational motion. In the chart on the previous page you should notice that torque is measured in Joules (Newton-metres), which is the same unit used to measure both work and energy. Does this mean that torque is the same as both energy and work? Before we come to a rash conclusion, we should remember that torque is a ***vector quantity*** while work and energy are both scalars. Nevertheless, the ***magnitude of torque*** must be measured in Joules because of the fact that we measure $|\vec{r}|$ in metres, $|\vec{F}|$ in Newtons and

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}|\sin\theta .$$

Now what does it mean that the torque of the 360 Modena's engine is 373 J? The images shown below should give you a good idea of how the ***pistons*** are connected to the ***crankshaft*** by means of ***connecting rods***.





In the diagram at the left, we can see how the force of the piston moving down is applied to the crankshaft. In this way, the up and down motion of the piston is converted into the rotational motion of the crankshaft.

Once we superimpose the vectors $\vec{r}$ and $\vec{F}$ onto the connecting rod and crankshaft respectively, we see that a torque is produced. Clearly, the torque is directed into the page when the piston is on its way down. When the piston is on its way back up, the torque is still directed into the page because the directions of both $\vec{r}$ and $\vec{F}$ are "reversed."

In the case of the Ferrari shown above, each piston produces a torque of about 46.6 J (373 ÷ 8).

Homework Exercises for Chapter 5 (Algebraic Vectors)

We have now reached the end of the second unit. You should now be ready to do the following:

1. Read all of chapter five once again. Pay attention to the examples in the textbook, especially those that are different from the examples found in these notes or those given in class.

2. Make summary notes of the ***main ideas of the unit***.
See page 14 for an example of how to create a good summary note.

3. Read these notes once again. Ensure that you understand all examples and that you have answered all questions.

4. Do the following homework sets. Whenever necessary, consult your textbook, these notes or any other resources.

Homework Set 1	***Homework Set 2***	***Homework Set 3***	***Homework Set 4***
p. 166 #1, 2c, 3c, 4d, 6	**p. 167** #7, 8, 9	**p. 168** #11, 12ef, 13ef, 14cd, 15	**p. 169** #23, 24, 25
p. 172 #1, 2hjl, 3bd, 4f, 5d	**p. 173** #6, 7cd, 8b, 9c, 10	**p. 173** #11, 12, 13, 14	**p. 174** #15, 16, 18, 19
p. 178 #1, 2, 3, 4, 5, 6, 7	**p. 179** #8, 9, 10, 11, 14	**p. 179** #15, 16, 17, 18, 19	**p. 180** #20, 21, 23, 24, 26
p. 185 #1, 2, 3, 4	**pp. 185-186** #5, 6, 7, 8, 9	**p. 186** #10cf, 11, 12	**p. 186** #13, 14, 15
p. 192 #1, 2, 3, 4	**p. 192** #5, 6, 7, 8, 9cd	**p. 193** #10, 11, 12, 14, 15d	**p. 193** #16, 17, 18, 19