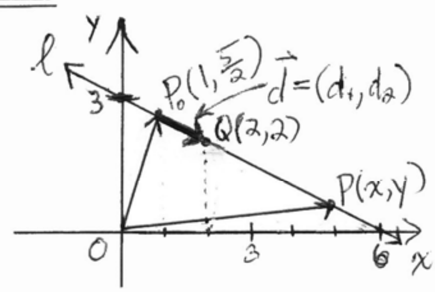
Unit 3 – Intersections of Lines and Planes

VECTOR EQUATIONS OF LINES IN $\mathbb{R}^2$

<u>Example</u>



- Let $P(x,y)$ represent ANY point on line ℓ.
- Let $\vec{d}=(d_1,d_2)$ represent a vector parallel to line ℓ (such a vector is called a <u>DIRECTION VECTOR for ℓ</u>)
- P_0 is any FIXED point on the line ℓ (in this case, P_0 has coordinates $(1,\frac{5}{2})$)

Notice that for any point $P(x,y)$ on ℓ,

$$\overrightarrow{OP}=\overrightarrow{OP_0}+t\vec{d}, \text{ where } t \text{ is some scalar.}$$

($\overrightarrow{OP_0}$ plus a certain scalar multiple of $\vec{d}$ is equal to $\overrightarrow{OP}$)

In this example,
$$\begin{aligned}\vec{d}&=\overrightarrow{P_0Q}\\&=\overrightarrow{OQ}-\overrightarrow{OP_0}\\&=(2,2)-(1,\tfrac{5}{2})\\&=(1,-\tfrac{1}{2})\end{aligned}$$

$\therefore \quad (x,y)=(1,\frac{5}{2})+t(1,-\frac{1}{2}),\ t\in\mathbb{R}$

An equation of this form is called a <u>VECTOR EQUATION</u> of a line. The variable "t" is called a <u>PARAMETER</u>.

<u>Note</u>: ① This kind of equation shows that a line is completely determined by a point and a direction vector

② With a vector equation, it is only necessary to choose values for the parameter "t" to obtain points on the line.

<u>Exercise</u>

Complete the given table of values

t	(x,y)
0	$(1,\frac{5}{2})$
1	
2	
-1	
-2	

By solving for x and y individually, we obtain the PARAMETRIC EQUATIONS of the line

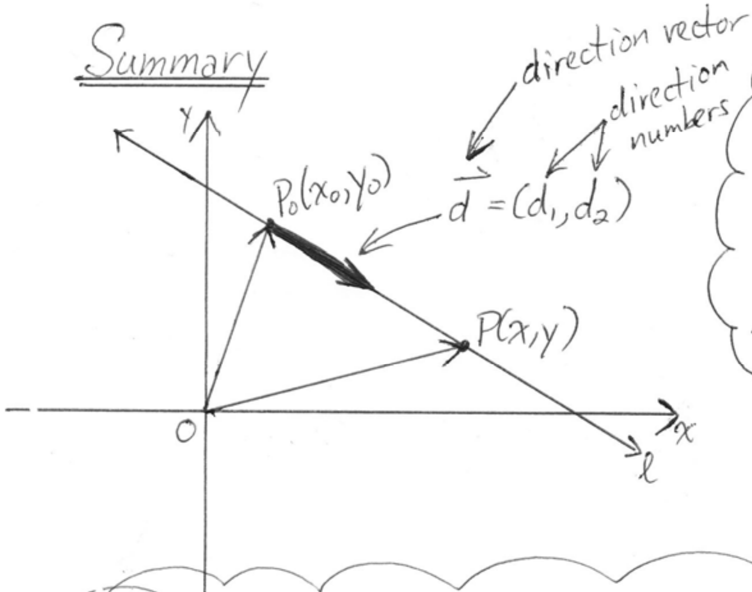
$$x = 1 + t$$
$$y = \frac{5}{2} - \frac{1}{2}t$$

By solving for the parameter t, we obtain the SYMMETRIC EQUATION of the line:

$t = x - 1$ and $t = \frac{y - \frac{5}{2}}{\left(-\frac{1}{2}\right)}$

$$\therefore \frac{x-1}{1} = \frac{y - \frac{5}{2}}{\left(-\frac{1}{2}\right)}$$

Summary



$\vec{OP}$ is equal to $\vec{OP_0}$ plus some scalar multiple of $\vec{d}$

$\therefore \vec{OP} = \vec{OP_0} + t\vec{d}, \quad t \in \mathbb{R}$

$\therefore (x, y) = (x_0, y_0) + t(d_1, d_2), \quad t \in \mathbb{R}$

A VECTOR EQUATION of ℓ → $(x, y) = (x_0, y_0) + t(d_1, d_2), \ t \in \mathbb{R}$

A SET OF PARAMETRIC EQUATIONS of ℓ → $\left.\begin{array}{l} x = x_0 + td_1 \\ y = y_0 + td_2 \end{array}\right.$

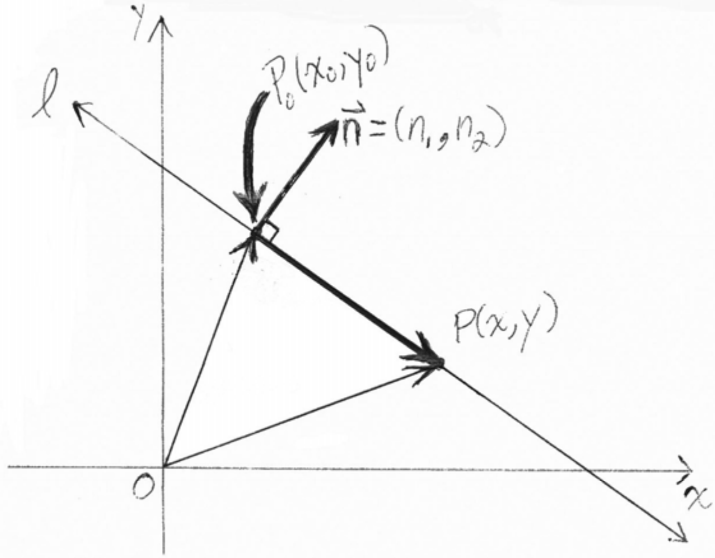
A SYMMETRIC EQUATION of ℓ → $\dfrac{x - x_0}{d_1} = \dfrac{y - y_0}{d_2}$

- (d_1, d_2) is any vector parallel to ℓ
- It's called a direction vector
- Its components are called direction numbers

Note Vector, parametric and symmetric equations depend on the choice of a point on the line and a direction vector for the line. Since there are an infinite number of such choices, there are an infinite number of vector equations (or parametric or symmetric) for any given line.

Homework: p. 245–247 # 2, 3, 4, 5, 8, 9, 11, 12, 15, 18

Scalar Equation of Lines in $\mathbb{R}^2$



$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$$
$$= (x, y) - (x_0, y_0)$$
$$= (x - x_0, y - y_0)$$

$\overrightarrow{P_0P}$ is a <u>direction vector</u> for ℓ

Let $\vec{n}$ be a <u>NORMAL VECTOR</u> to ℓ $(\vec{n} \perp \ell)$.

$$\because \vec{n} \perp \ell$$
$$\therefore \vec{n} \perp \overrightarrow{P_0P}$$
$$\therefore \vec{n} \cdot \overrightarrow{P_0P} = 0$$
$$\therefore (n_1, n_2) \cdot (x - x_0, y - y_0) = 0$$
$$\therefore n_1(x - x_0) + n_2(y - y_0) = 0$$
$$\therefore n_1 x - n_1 x_0 + n_2 y - n_2 y_0 = 0$$
$$\therefore n_1 x + n_2 y - n_1 x_0 - n_2 y_0 = 0$$

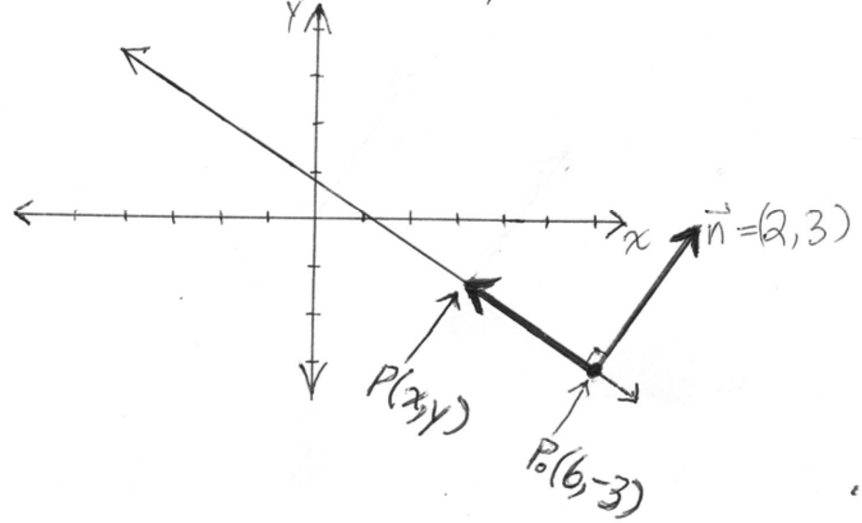
If we let $A = n_1$, $B = n_2$ and $C = -n_1 x_0 - n_2 y_0$, we obtain the familiar Cartesian (scalar) equation of a line:

$$Ax + By + C = 0$$

<u>Note:</u> If an equation of a line is given in the form $Ax + By + C = 0$, then the vector (A, B) is normal (perpendicular) to the line.

Example Find a scalar (Cartesian) equation of the line through $P_0(6,-3)$ with normal $\vec{n}=(2,3)$.

Solution 1 (First Principles)



$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$
$= (x,y)-(6,-3)$
$= (x-6, y+3)$

$\because \vec{n} \perp \overrightarrow{P_0P}$
$\therefore \vec{n} \cdot \overrightarrow{P_0P} = 0$
$\therefore (2,3)\cdot(x-6, y+3) = 0$
$\therefore 2x-12+3y+9=0$
$\therefore 2x+3y-3=0$ is a scalar equation of the line //

Solution 2 (A Much Easier Method)

$\because \vec{n}=(2,3)$ is normal to the line
$\therefore$ a Cartesian equation of the line is of the form

$$2x+3y+C=0$$

$\because (6,-3)$ lies on the line
$\therefore 2(6)+3(-3)+C=0$

$\therefore 3+C=0$
$\therefore C=-3$

$\therefore 2x+3y-3=0$ is a Cartesian equation of the line. //

Investigation

Let $P_0(x_0,y_0,z_0)$ be a point in $\mathbb{R}^3$ and let $P(x,y,z)$ represent a general point in $\mathbb{R}^3$.

Let $\vec{n}=(n_1,n_2,n_3)$ be perpendicular to $\overrightarrow{P_0P}=(x-x_0, y-y_0, z-z_0)$.

Then $\vec{n}\cdot\overrightarrow{P_0P}=0$.

(a) What equation is generated by using $\vec{n}\cdot\overrightarrow{P_0P}=0$?

(b) Does the equation generate a line in $\mathbb{R}^3$ or some other geometric object? Explain.

Homework: pp. 251–252 #1, 2, 3d, 4, 5, 6cd, 7b, 9b, 10

As we have already learned, there is no such thing as a Cartesian equation of a line in $\mathbb{R}^3$. A Cartesian equation in $\mathbb{R}^3$ describes a PLANE.

Therefore, to describe lines in $\mathbb{R}^3$ we ARE FORCED TO RESORT to VECTOR METHODS. Since we have already developed vector equations for lines in $\mathbb{R}^2$, it is only necessary to state the results for $\mathbb{R}^3$.

VECTOR EQUATIONS FOR LINES IN $\mathbb{R}^3$

Let ℓ represent any line in $\mathbb{R}^3$. If $\vec{d} = (d_1, d_2, d_3)$ is a direction vector for ℓ and $P_0(x_0, y_0, z_0)$ is a point on ℓ, then $(t \in \mathbb{R})$

$$\left.\begin{aligned} &(1)\quad \overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{d} \\ &(2)\quad (x, y, z) = (x_0, y_0, z_0) + t(d_1, d_2, d_3) \end{aligned}\right\} \text{Vector Equations}$$

$$(3)\quad \left.\begin{aligned} x &= x_0 + td_1 \\ y &= y_0 + td_2 \\ z &= z_0 + td_3 \end{aligned}\right\} \text{Parametric Equations}$$

$$(4)\quad \frac{x - x_0}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3} \quad \text{Symmetric Equations } (d_1 \neq 0,\ d_2 \neq 0,\ d_3 \neq 0)$$

Example

Do the equations $(x,y,z)=(2,0,9)+r(-1,5,2)$ and $(x,y,z)=(3,1,1)+s(1,-5,-2)$ $\Big\}\ r\in\mathbb{R}, s\in\mathbb{R}$

describe the same line?

Solution: $\because$ the direction vectors $(-1,5,2)$ and $(1,-5,-2)$ are parallel, the two equations MIGHT represent the same line.

Now rewrite one of the equations in parametric form:

$$x = 3+s$$
$$y = 1-5s$$
$$z = 1-2s$$

We know that $(2,0,9)$ lies on the first line. If it lies on the second line, it must satisfy the above parametric equations:

$$2 = 3+s \Rightarrow s=-1$$
$$0 = 1-5s \Rightarrow s=\tfrac{1}{5}$$
$$9 = 1-2s \Rightarrow s=-4$$

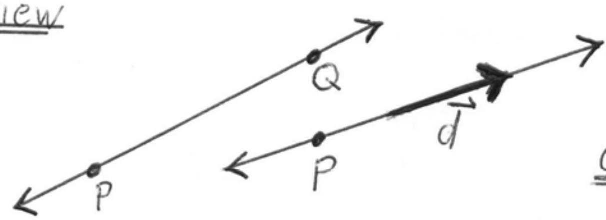
Since it is impossible for s to simultaneously equal two or more different values, we must conclude that $(2,0,9)$ DOES NOT lie on the second line.

Therefore, the two equations cannot describe the same line.

Homework: pp. 256-258 #1, 2c, 3c, 4, 6, 7, 8, 9, 10, 11, 12, 14, 15

Vector and Scalar Equations of Planes

Review



A line is completely determined by

① Two distinct points

OR ② A point and a direction vector

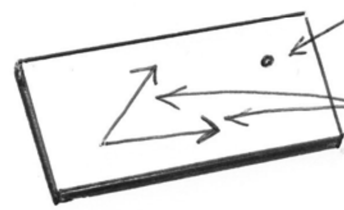
Planes

A plane is completely determined by

① Three non-collinear points that lie on the plane

OR

② One point on the plane and Two non-collinear (linearly independent) direction vectors



fixes location of plane

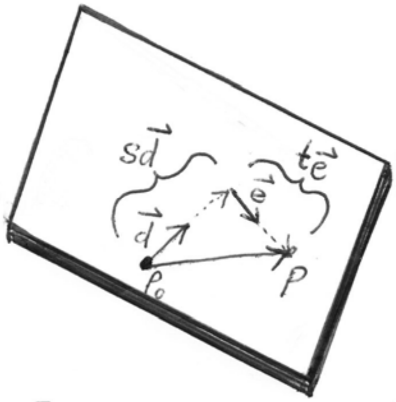
determine the "slant" of the plane

Vector Equations of Planes

Let $\vec{d}$ and $\vec{e}$ be non-collinear (linearly independent) vectors and let $P_0(x_0, y_0, z_0)$ represent a point. Also, let $P(x, y, z)$ represent ANY point on the plane determined by $\vec{d}$, $\vec{e}$ and P_0.

$\because$ $\overrightarrow{P_0P}$ lies on the plane and $\vec{d}$ and $\vec{e}$ are linearly independent, $\overrightarrow{P_0P}$ can be written as a linear combination of $\vec{d}$ and $\vec{e}$

i.e. $\overrightarrow{P_0P} = s\vec{d} + t\vec{e}$



$\therefore \overrightarrow{P_0P} = s\vec{d} + t\vec{e}$

$\therefore \overrightarrow{OP} - \overrightarrow{OP_0} = s\vec{d} + t\vec{e}$

$\therefore \overrightarrow{OP} = \overrightarrow{OP_0} + s\vec{d} + t\vec{e}$

$\therefore (x,y,z) = (x_0,y_0,z_0) + s(d_1,d_2,d_3) + t(e_1,e_2,e_3)$

Summary

A vector equation of the plane through $P_0(x_0,y_0,z_0)$ and with non-collinear direction vectors $\vec{d}$ and $\vec{e}$ is $(s \in \mathbb{R},\ t \in \mathbb{R})$

$$\overrightarrow{OP} = \overrightarrow{OP_0} + s\vec{d} + t\vec{e}$$

or

$$(x,y,z) = (x_0,y_0,z_0) + s(d_1,d_2,d_3) + t(e_1,e_2,e_3)$$

Parametric Equations

By solving for x, y and z, we obtain the parametric equations $(s \in \mathbb{R},\ t \in \mathbb{R})$

$$x = x_0 + sd_1 + te_1$$
$$y = y_0 + sd_2 + te_2$$
$$z = z_0 + sd_3 + te_3$$

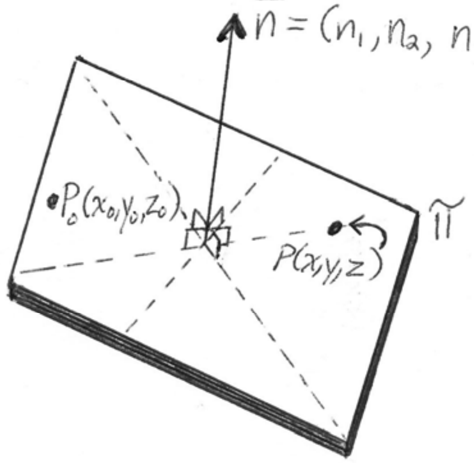
Questions

1. Explain why it is not possible to write symmetric equations of planes in $\mathbb{R}^3$.

2. Explain a procedure for finding Cartesian (scalar) equations of planes.

Cartesian (Scalar) Equations of Planes

Let π represent the plane shown at the right and let $\vec{n}$ represent a vector perpendicular to the plane (i.e. a normal vector).



Since $\vec{n}$ is a normal vector, $\vec{n}$ is perpendicular to any vector on π and to any line on π.

Since P_0 and P both lie on the plane π,

$\therefore \ \overrightarrow{P_0P}$ lies on π

$\because \ \vec{n}$ is perpendicular to every vector on π

$\therefore \ \vec{n} \perp \overrightarrow{P_0P}$

$\therefore \ \vec{n} \cdot \overrightarrow{P_0P} = 0$

$\therefore \ (n_1, n_2, n_3) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$\therefore \ n_1 x - n_1 x_0 + n_2 y - n_2 y_0 + n_3 z - n_3 z_0 = 0$

$\therefore \ n_1 x + n_2 y + n_3 z - n_1 x_0 - n_2 y_0 - n_3 z_0 = 0$

If we let $A = n_1$, $B = n_2$, $C = n_3$ and $D = -n_1 x_0 - n_2 y_0 - n_3 z_0$, we obtain the familiar equation

$$Ax + By + Cz + D = 0.$$

However, THIS IS NOT AN EQUATION OF A LINE. Since the points P_0 and P can be anywhere on π, $Ax + By + Cz + D = 0$ DESCRIBES A PLANE!!

Summary

An equation of the form $Ax+By+Cz+D=0$ describes a PLANE in $\mathbb{R}^3$. The vector (A,B,C) is a normal for the plane. Such an equation is called a CARTESIAN or SCALAR EQUATION of a plane.

Example

Rewrite the equation
$(x,y,z)=(0,1,3)+s(-2,3,1)+t(1,0,1)$
as a scalar equation of a plane.

Solution 1

$\because$ $(-2,3,1)$ and $(1,0,1)$ are direction vectors for the plane,

$\vec{n}=(-2,3,1)\times(1,0,1)$
$=(3,3,-3)$ is a normal for the plane

$\therefore$ a Cartesian equation of the plane is of the form

$$3x+3y-3z+D=0$$

$\because$ $(0,1,3)$ lies on the plane

$3(0)+3(1)-3(3)+D=0$

$\therefore$ $D=6$

$\therefore$ $3x+3y-3z+6=0$ is a scalar equation of the plane (dividing both sides by 3 yields the equation $x+y-z+2=0$)

Solution 2

As shown in solution 1, $\vec{n}=(3,3,-3)$ is a normal for the plane.

Let $P(x,y,z)$ represent any point on the plane and P_0 represent $(0,1,3)$, which lies on the plane.

Then $\overrightarrow{P_0P}\perp n$

$\therefore$ $\vec{n}\cdot\overrightarrow{P_0P}=0$

$\therefore$ $(3,3,-3)\cdot(x,y-1,z-3)=0$

$\therefore$ $3x+3y-3-3z+9=0$

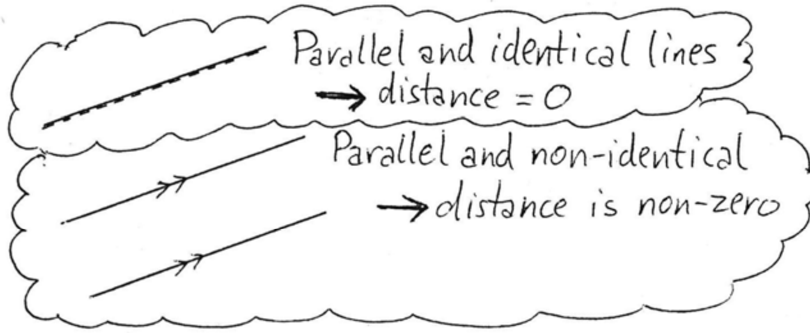
$\therefore$ $3x+3y-3z+6=0$ is a scalar equation of the plane

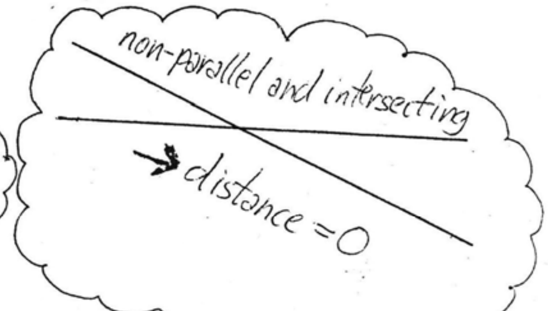
Homework

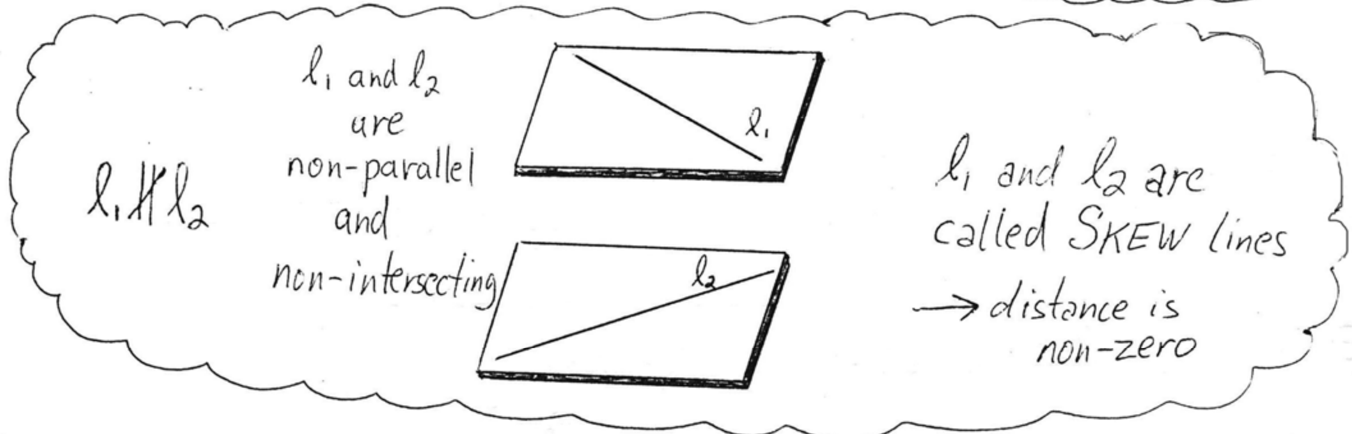
pp. 279–281
#1, 2, 3, 4b, 5bc, 6bc, 7bde, 8ace, 10, 11, 13, 14

pp. 285–288
#2, 4, 5ef, 6b, 7bd, 8, 9, 10, 11

Calculating Distances from Points/Lines/Planes to Points/Lines/Planes



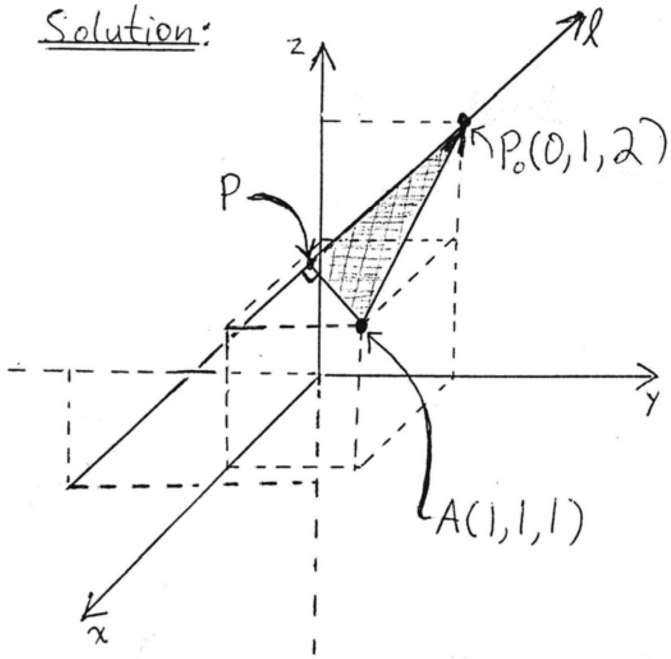




Example 1 – Distance from a Point to a Line

Find the distance from $A(1,1,1)$ to the line l: $(x,y,z)=(0,1,2)+t(0,1,1)$

Solution:



Let $\vec{d}=(0,1,1)$. Because of the equation given for l, we know that $\vec{d}$ is a direction vector for l.

Then $|\overrightarrow{P_0P}| = |\text{proj}_{\vec{d}}\overrightarrow{AP_0}|$

$$= \frac{|\overrightarrow{AP_0}\cdot\vec{d}|}{|\vec{d}|}$$

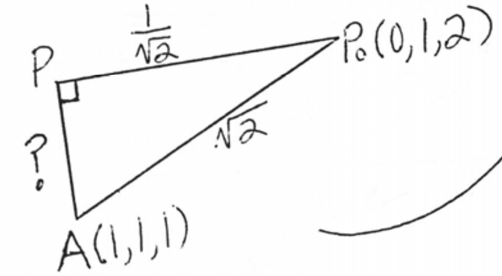
absolute value

magnitude

$$= \frac{|(-1,0,1)\cdot(0,1,1)|}{\sqrt{0^2+1^2+1^2}}$$

$$= \frac{1}{\sqrt{2}}$$

Also $|\overrightarrow{AP_0}| = \sqrt{(-1)^2+0^2+1^2}$
$= \sqrt{2}$



$\therefore$ distance from A to ℓ

$= |\overrightarrow{AP}|$

$= \sqrt{|\overrightarrow{AP_0}|^2 - |\overrightarrow{P_0P}|^2}$

$= \sqrt{(\sqrt{2})^2 - \left(\frac{1}{\sqrt{2}}\right)^2}$

$= \sqrt{2 - \frac{1}{2}}$

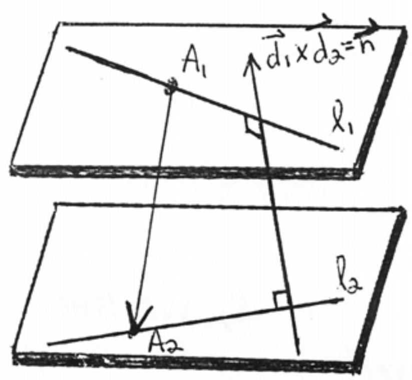
$= \sqrt{\frac{3}{2}}$

Example 2 – Distance between Two Parallel Lines

Find the distance the lines ℓ_1 and ℓ_2 with vector equations
$(x,y,z) = (4,-7,1) + s(3,0,2)$ and $(x,y,z) = (1,4,5) + (3,0,2)t$

Solution: Choose a point on ℓ_1 and then find the distance from that point to ℓ_2 (or vice versa) using the procedure of example 1. //

Example 3 – Distance between Skew Lines



$\vec{d_1}$ – direction vector for ℓ_1

$\vec{d_2}$ – direction vector for ℓ_2

$\therefore \vec{d_1} \times \vec{d_2} \perp \ell_1$ and $\vec{d_1} \times \vec{d_2} \perp \ell_2$

Let A_1 represent any point on ℓ_1 and let A_2 represent any point on ℓ_2.

Then $D = |\text{proj}_{\vec{d_1} \times \vec{d_2}} \overrightarrow{A_1A_2}|$

$= |\text{proj}_{\vec{n}} \overrightarrow{A_1A_2}|$ (since $\vec{n} = \vec{d_1} \times \vec{d_2}$)

= distance between skew lines ℓ_1 and ℓ_2

For lines ℓ_1 and ℓ_2 given by

ℓ_1: $x-1 = y = z$ $\quad$ ℓ_2: $\frac{x}{2} = 1-y = z$

(a) show that ℓ_1 and ℓ_2 are skew lines
(b) find the distance between ℓ_1 and ℓ_2

Solution: (a) $\because$ the direction vectors $\vec{d_1} = (1,1,1)$ and $\vec{d_2} = (2,-1,1)$ are NOT parallel

$\therefore \ell_1 \nparallel \ell_2$ (continued →)

To show that l_1 and l_2 Do NOT intersect, we attempt to find the intersection of l_1 and l_2.

Rewrite each equation in parametric form:

$$l_1: \begin{aligned} x &= 1+t \\ y &= t \\ z &= t \end{aligned} \qquad l_2: \begin{aligned} x &= 2s \\ y &= 1-s \\ z &= s \end{aligned}$$

If there were a point of intersection, at the point of intersection

$$1+t = 2s, \quad t = 1-s, \quad t = s$$

$$\therefore \begin{aligned} 2s - t &= 1 \quad (1) \\ s + t &= 1 \quad (2) \\ s - t &= 0 \quad (3) \end{aligned}$$

$(2) + (3)$ $\quad 2s = 1$

$$\therefore s = \tfrac{1}{2}$$

Substitute in (3) to obtain

$$t = \tfrac{1}{2}$$

Substituting in (1) we obtain

$$L.S. = 2(\tfrac{1}{2}) - \tfrac{1}{2} = 1 - \tfrac{1}{2} = \tfrac{1}{2} \neq 1 = R.S.$$

$\therefore$ l_1 and l_2 do not intersect and $l_1 \nparallel l_2$

$\therefore$ l_1 and l_2 are skew lines

(b) $\vec{d_1} = (1,1,1)$ and $\vec{d_2} = (2,-1,1)$ are direction vectors for l_1 & l_2 respectively

$\therefore$ $\vec{n} = \vec{d_1} \times \vec{d_2} = (2,1,-3)$ is perpendicular to l_1 and l_2

$A_1(1,0,0)$ lies on l_1 and $A_2(0,1,0)$ lies on l_2

$\therefore$ distance between l_1 and l_2

$$= \left|\text{proj}_{\vec{n}} \overrightarrow{A_1A_2}\right|$$

$$= \frac{|\overrightarrow{A_1A_2} \cdot \vec{n}|}{|\vec{n}|}$$

absolute value

magnitude

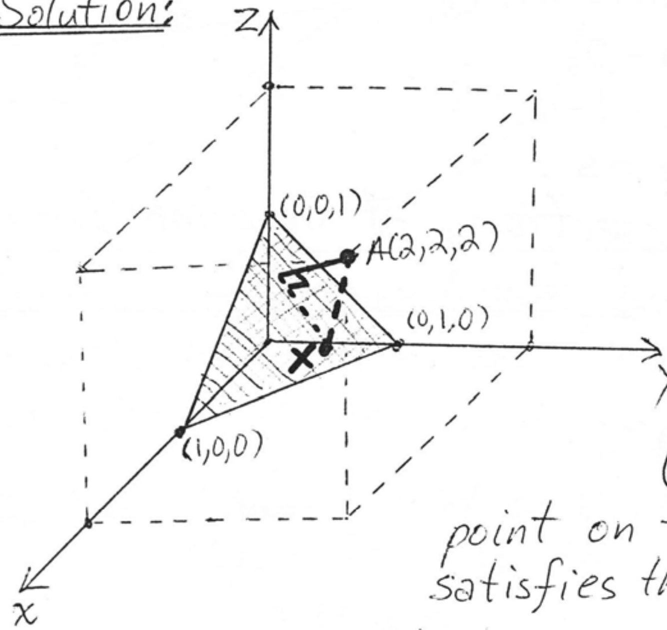
$$= \frac{1}{\sqrt{14}}$$

$$\doteq 0.3$$

Example 4 – Distance between a Point and a Plane

Find the distance from $A(2,2,2)$ to the plane $x+y+z=1$.

Solution:



The plane $x+y+z=1$ has x-intercept 1, y-intercept 1 and z-intercept 1. (A triangular PORTION of the plane is shown in the diagram.)

Let X represent the point $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ (X can be any point on the plane, i.e. any point that satisfies the equation).

Then $\overrightarrow{AX} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) - (2,2,2)$

Remember that these are position vectors NOT points!

$$= \left(\frac{-7}{4}, \frac{-3}{2}, \frac{-7}{4}\right)$$

Also, $\vec{n} = (1,1,1)$ is a normal for the plane.

The distance from A to the plane is the projection of $\overrightarrow{AX}$ onto any vector perpendicular to the plane.

$$\therefore \quad \text{distance} = \left|\operatorname{proj}_{\vec{n}} \overrightarrow{AX}\right|$$
$$= \frac{|\overrightarrow{AX} \cdot \vec{n}|}{|\vec{n}|}$$
$$= \frac{\left|-\frac{7}{4} - \frac{3}{2} - \frac{7}{4}\right|}{\sqrt{1^2+1^2+1^2}}$$
$$= \frac{5}{\sqrt{3}}$$

Homework

- p.252 # 12 p.258 # 15
- Questions 1, 3, 4, 5, 6 on the next page
- p.287 # 15
- p.313 # 11

B 1. Find a normal for each line.
(a) $2x - 5y = 1$
(b) $(x-2, y) = r(3, -1)$
(c) $\overrightarrow{OP} = (1, 0, -1) + s(2, 0, 1)$
(d) $\dfrac{x-2}{3} = 2 - y = \dfrac{z+1}{5}$

2. Calculate the cross product $\vec{u} \times \vec{v}$ for the given vectors.
(a) $\vec{u} = (1, 0, 5)$, $\vec{v} = (-2, 7, 1)$
(b) $\vec{u} = (2, 3, -1)$, $\vec{v} = (0, 2, 5)$

3. Find the distance from the point $(0, 4, 1)$ to each of the following lines.
(a) the line $\overrightarrow{OP} = (1, 0, 5) + r(-1, 0, 1)$
(b) the line $x = 4 - 2s$, $y = 3 + s$, $z = 5s$
(c) the line $\dfrac{1-x}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4}$

4. The method of Example 1 can be applied to find the distance from a point to a line in the plane.
(a) Give the coordinates of one point, call it point X, on the line l: $x = 3 + s$, $y = 1 - 3s$.
(b) For the point $A(1, -2)$ find the magnitude of the projection of $\overrightarrow{AX}$ on the direction vector $(1, -3)$ of the line l.
(c) Draw a graph showing points A and X, the line l, and the projection of $\overrightarrow{AX}$ on l.
(d) On the graph label the perpendicular distance from A to l as D.
(e) Use the Pythagorean Theorem to calculate D, the distance from the point A to the line l.

5. Using the method of Question 4, find the distance from the point $(-1, 5)$ to each line in the plane.
(a) the line $x = 2 - 5t$, $y = 3t$
(b) the line $\overrightarrow{OP} = (4, 2) + r(-1, 1)$
(c) the line $\dfrac{x-2}{3} = 4 - y$

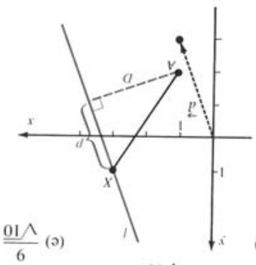
6. It is also possible to find the distance from a point to a line in the same plane using a normal to the line and a point on the line.
(a) Find a normal $\vec{n}$ to the line $2x - 3y = 5$.
(b) Choose one point, call it X, on the line $2x - 3y = 5$.
(c) Draw a graph showing the line $2x - 3y = 5$, its normal $\vec{n}$, the point X, and the point $A(3, 1)$.
(d) Find the magnitude of the projection of $\overrightarrow{AX}$ on $\vec{n}$.
(e) What is the connection between the quantity found in (d) and the perpendicular distance from the point A to the line l?

Answers to Questions 3 to 6

3. (a) $\sqrt{\frac{57}{2}} \doteq 5.3$ (b) $\sqrt{\frac{173}{15}} \doteq 3.4$ (c) $\sqrt{\frac{465}{29}} \doteq 4.0$

4. (a) $X(3, 1)$ (b) $\dfrac{7}{\sqrt{10}}$

(c) (d)

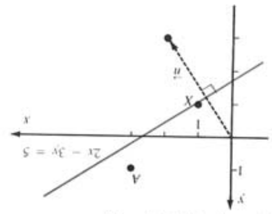


(e) $\dfrac{9}{\sqrt{10}} \doteq 2.8$

5. (a) $\dfrac{16}{\sqrt{34}} \doteq 2.7$ (b) $\sqrt{2} \doteq 1.4$ (c) 0

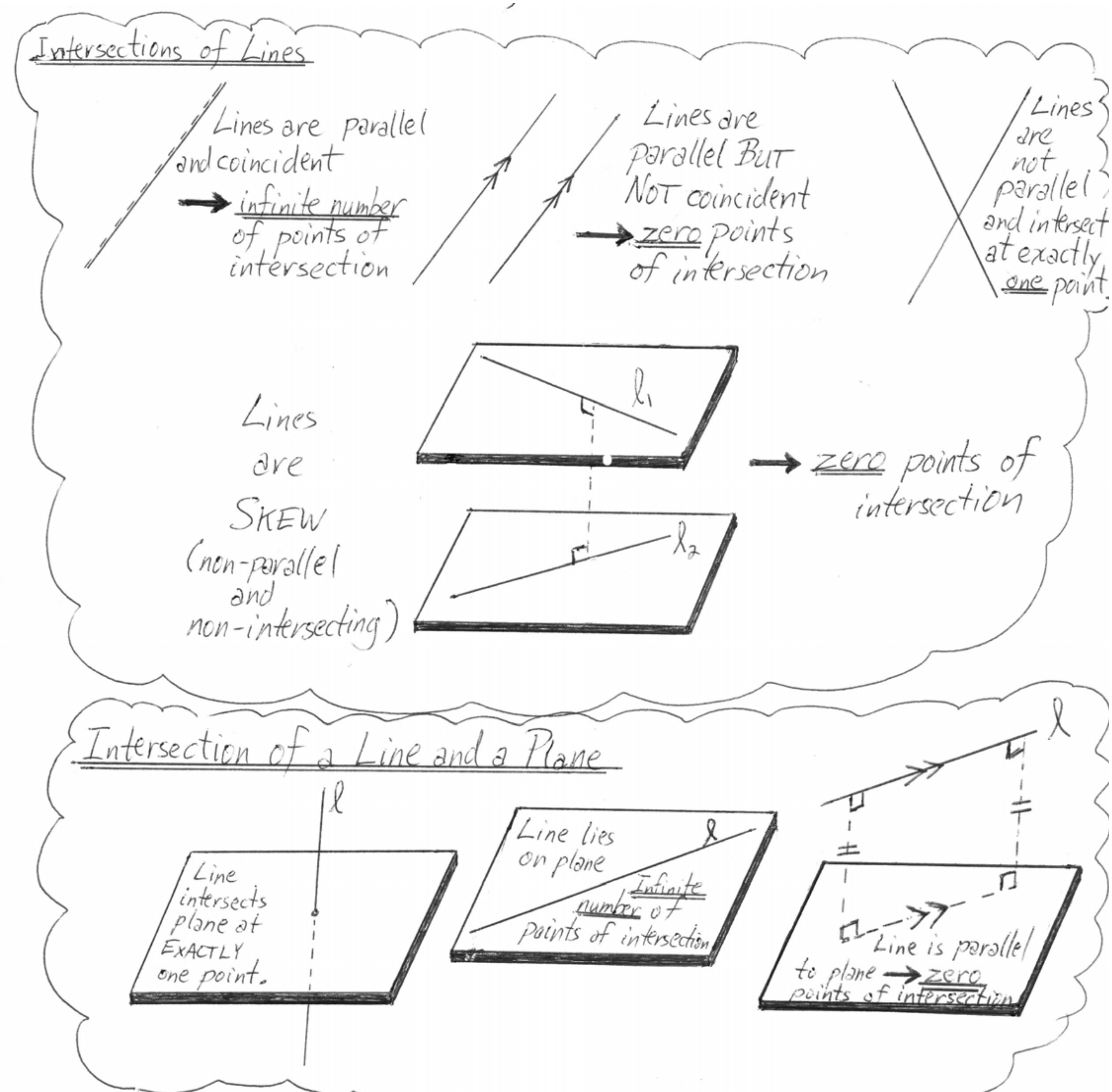
6. (a) $(2, -3)$ (b) $X(1, -1)$

(c)



(d) $\dfrac{2}{\sqrt{13}}$ (e) same

Intersections of Lines and Intersections of Lines and Planes



Example 1 Investigate the point(s) of intersection of the lines l_1 and l_2 for $l_1: (x,y,z) = (-2,1,0) + s(1,3,7)$

$l_2: \frac{x-3}{5} = \frac{y+3}{-4} = \frac{4-z}{2}$

Solution:

Example 2

(a) Show that there is no point of intersection of the line $\frac{x-2}{3} = \frac{y+5}{1} = \frac{z-6}{8}$ and the plane $5x+y-2z+2=0$

Solution:

(b) Find the point(s) of intersection (if **any**) of the line $\frac{x-2}{1} = \frac{y+1}{-2} = \frac{z+3}{-5}$ and the plane $3x+19y-7z-8=0$

Solution:

Homework: p. 263 # 3, 4cde, 5, 7, 9, 10, 15, 16
p. 292 # 1de, 2, 3, 4, 6, 8, 9cd, 10aef, 11, 12

A Little bit of Vector Theory that Helps Analyze the Intersection of Planes

Linear Combination of Vectors

If $\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n$ then $\vec{v}$ is said to be a ***linear combination*** of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$. For example, the vector $\vec{v} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ is a linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Linear Independence of Vectors

The vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ are said to be ***linearly independent*** if the only linear combination of the vectors that produces the zero vector ($\vec{0}$) is $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_n$. That is, if $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n = \vec{0}$, then $a_1 = 0, a_2 = 0, \ldots, a_n = 0$. Vectors that are ***not*** linearly independent are said to be ***linearly dependent***.

Linear Dependence and Linear Independence in $\mathbb{R}^2$ and $\mathbb{R}^3$

Vector Space	Maximum Number of Linearly Independent Vectors in any Set of Vectors	Geometric Significance	Diagrams
$\mathbb{R}^2$	2	• Linearly independent vectors are ***non-parallel*** • Linearly dependent vectors are ***parallel***	
$\mathbb{R}^3$	3	• Linearly independent vectors are ***non-coplanar*** • Linearly dependent vectors are ***coplanar***	

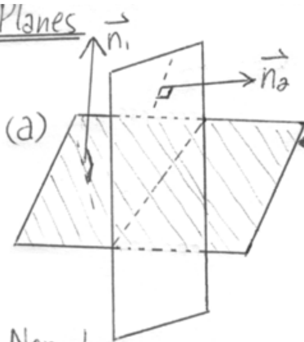
Simple Test for Linear Independence in $\mathbb{R}^3$

The vectors $\vec{u}, \vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ are ***linearly independent*** (non-coplanar) if and only if $\vec{u} \times \vec{v} \cdot \vec{w} \neq 0$. That is, the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ are ***non-coplanar*** if and only if the ***triple scalar product is non-zero***. (Alternatively, $\vec{u}, \vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ are ***coplanar*** if and only if $\vec{u} \times \vec{v} \cdot \vec{w} = 0$.)

Proof: (This proof is left up to you. It is straightforward if you understand the dot product and cross product.)

Intersections of Planes and Systems of Linear Equations

Two Planes



Normals are NOT parallel (i.e. NOT collinear)

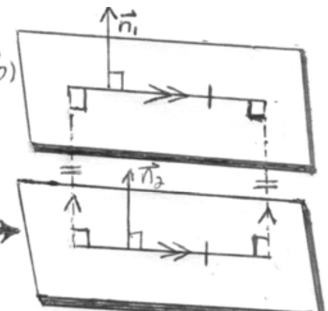
(a) Two planes either intersect in a line

OR

(b) they are parallel and distinct (they do not intersect)

OR

(c) they are parallel and coincident (they intersect at an infinite number of points) → (no diagram)

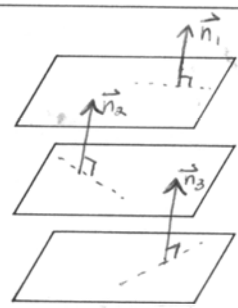


Normals are parallel (i.e. collinear)

Three Planes

Type I

If $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$, are collinear, the planes are parallel, and, being distinct, do not intersect.

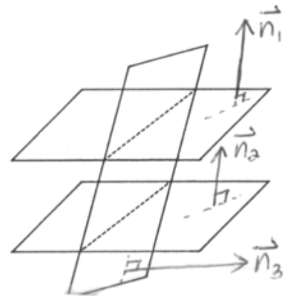


Type I
All planes are parallel.

Type II
Only two planes are parallel.

Type II

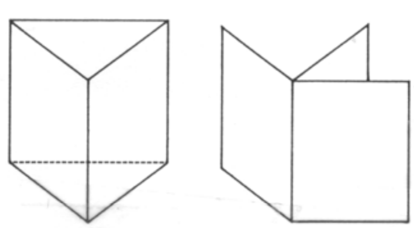
If only one pair of $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ is collinear then there is a pair of parallel planes.



Type III

If the normals $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ are coplanar but no pair is collinear then either the planes have no point in common, or they have a line in common.

Recall...
$\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar iff $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$

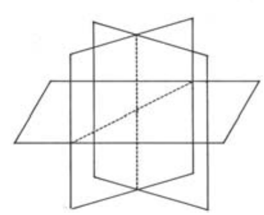


Type III
No planes are parallel but normals are coplanar → $\therefore (\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 = 0$

Type IV
Normals not coplanar → $(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n}_3 \neq 0$

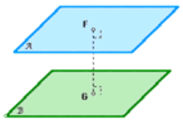
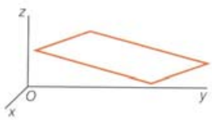
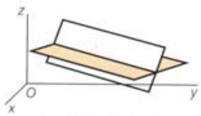
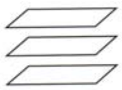
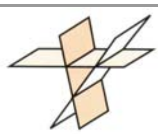
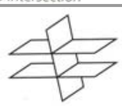
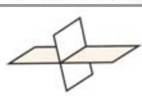
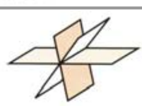
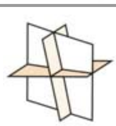
Type IV

If $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ are not coplanar then the planes intersect in a single point.



Important Exercise

Complete the following table. Two rows have been done for you.

	Diagram	*Type of Intersection*	*Nature of Normals*	*Example Linear System*
Two Planes		No intersection. The planes are parallel and distinct.	The normals are parallel but the planes do not have any common points.	$x+2y+3z=4$ (1) $2x+4y+6z=9$ (2)
	 planes are coincident			
	 planes intersect in a line			
Three Planes	 2 planes are coincident, the other parallel; no intersection			
	 3 planes are parallel and distinct; no intersection			
	 normals coplanar; no intersection			
	 two planes are parallel and distinct, the other crossing; no common intersection			
	 3 planes are coincident; intersection: a plane			
	 two planes are coincident; the other crossing; intersection: a line			
	 normals coplanar; intersection: a line			
	 normals are not parallel and non-coplanar; intersection: a point	The three planes intersect at a single point (type IV).	The normals are ***non-coplanar***. Therefore, the triple scalar product of the normals is non-zero. $\vec{n}_1 \times \vec{n}_2 \cdot \vec{n}_3 \neq 0$	$x+y-3=0$ (1) $y+z+5=0$ (2) $x+z+2=0$ (3)

System of m linear equations in n unknowns
- Inconsistent (no solution)
- Consistent (at least 1 solution)
  - Independent (unique solution)
  - Dependent (more than one solution)

Examples Solve and classify each system of equations. Before applying the method of elimination, use information about the normal vectors to predict the type of intersection (solution).

(a)
$$x+y-3=0 \quad ①$$
$$y+z+5=0 \quad ②$$
$$x+z+2=0 \quad ③$$

Solution:

(i) Prediction using Normals

$\vec{n}_1=(1,1,0)$, $\vec{n}_2=(0,1,1)$, $\vec{n}_3=(1,0,1)$

$(\vec{n}_1\times\vec{n}_2)\cdot\vec{n}_3 = 2 \neq 0$ ∴ planes intersect at one point ∴ system is CONSISTENT and INDEPENDENT

(ii) Method of Elimination

② − ③, $y-x+3=0$ ④

④ + ①, $2y=0$

$\therefore y=0$

Substituting in ①, we obtain

$x-3=0$

$\therefore x=3$

Substituting in ③, we obtain

$z+5=0$

$\therefore z=-5$

$\therefore x=3,\ y=0,\ z=-5$

(continued next page) →

(b)
$$x+2y+3z-4=0 \quad ①$$
$$2x-y+4z+7=0 \quad ②$$
$$3x-14y+z+48=0 \quad ③$$

Solution:

(i) Prediction using Normals

(ii) Method of Elimination

(c)
$$x+2y+3z-4=0 \quad ①$$
$$2x-y+4z+7=0 \quad ②$$
$$3x-14y+z-20=0 \quad ③$$

Solution:

(i) Prediction using Normals

(ii) Method of Elimination

$\therefore (x, y, z) = (3, 0, -5)$ is the only solution

$\therefore$ the system is CONSISTENT AND INDEPENDENT

(d) $x + 2y = 4$ ①
$3x + 6y = 11$ ②

Solution:

(i) Prediction using Normals

$\vec{n}_1 = (1, 2)$, $\vec{n}_2 = (3, 6) = 3\vec{n}_1$

$\therefore \vec{n}_1 \parallel \vec{n}_2$, but $11 \neq 3(4)$

$\therefore$ lines are parallel and distinct

$\therefore$ system is INCONSISTENT

(ii) Method of Elimination

$3 \times$①, $3x + 6y = 12$ ③

③ − ②, $0 = 1$, which is absurd. Therefore, the system is INCONSISTENT.

(e) $x + 2y = 4$ ①
$3x + 6y = 12$ ②

Solution:

(i) Prediction using Normals

(ii) Method of Elimination

(f) $4x - 5y - 2z - 1 = 0$ ①
$x - y - 2z = 0$ ②

Solution:

(i) Prediction using Normals

(ii) Method of Elimination

(g) $x + 2y = 4$ ①
$3x + 5y = 11$ ②
$x - y = 1$ ③

Solution:

(i) Prediction using Normals

(ii) Method of Elimination

(h) $x + 2y = 4$ ①
$3x + 5y = 11$ ②
$2x + 4y = 10$ ③

Solution:

(i) Prediction using Normals

(ii) Method of Elimination

Homework: pp. 300 – 301 #1, 2, 3 pp. 308 – 309 #2, 3, 4, 9

Using Matrices to Perform Gaussian Elimination and Gauss-Jordan Elimination

Introduction

Although the method of elimination of the previous section works very well, it tends to be long and tedious. In particular, it is extremely tiresome having to copy all the literal coefficients (i.e. the x's, y's and z's) from one line to the next. Since mathematicians always strive to strip away all but the essential details, a method has been devised that allows us to solve systems of linear equations ***without*** having to write the literal coefficients at all! This approach requires the use of ***matrices***, which are introduced in the next section.

What is a Matrix?

A ***matrix*** (plural ***matrices***) is a rectangular array of ***elements*** (or ***entries***) set out by rows and columns. Matrices are used to store numbers (or any other mathematical objects) in rows and columns for a variety of different applications including transformations, graph theory and solving systems of equations. The diagrams below should help to bring this rather abstract discussion into the realm of tangibility.

An Example of a 4×3 (4 by 3) Matrix
(4 rows, 3 columns)

$$\begin{array}{c c} & \begin{array}{ccc}\text{Column 1} & \text{Column 2} & \text{Column 3}\end{array} \\ \begin{array}{c}\text{Row 1}\\ \text{Row 2}\\ \text{Row 3}\\ \text{Row 4}\end{array} & \begin{pmatrix} 2 & 4 & -1 \\ -2 & 0 & 0 \\ 8 & 9 & -7 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Notation used to Identify Entries (Elements) of a Matrix

Entry (Element) in Row 2, Column 3

Entry (element) in row 4, column 2

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

Entry (element) in row 2, column 1

In general, an "m by n" matrix (written $m \times n$) consists of m rows and n columns. In any matrix A, the entry (element) found in row r and column s is denoted a_{rs}.

How are Matrices used to Solve Systems of Linear Equations?

Consider the following system of linear equations:

$$x + 2y + 3z = 4 \quad (1)$$

$$2x - y + 4z = -7 \quad (2)$$

$$3x - 14y + z = -48 \quad (3)$$

Prediction using Normal Vectors of the Planes Corresponding to the Equations

There is no pair of parallel normals. Thus, the system may have a ***unique solution***, ***an infinite number of solutions*** or ***no solution***.

$$\begin{aligned} & n_1 \times n_2 \cdot n_3 \\ &= (1,2,3) \times (2,-1,4) \cdot (3,-14,1) \\ &= (11,2,-5) \cdot (3,-14,1) \\ &= 0 \end{aligned}$$

Since the triple scalar product is zero, the normal vectors must be ***coplanar*** (linearly dependent).
Therefore, the system has ***no solutions*** or ***an infinite number of solutions*** (Type III).

The first step in using a matrix to solve such a system is to write the ***augmented matrix*** of the system. The augmented matrix of a linear system consists entirely of the ***numerical coefficients*** of the system, written in the same order as they appear in the equations. Thus, the augmented matrix for the above system is written as follows:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & 4 & -7 \\ 3 & -14 & 1 & -48 \end{array}\right)$$

Note that the first column contains the numerical coefficients of x, the second column contains the numerical coefficients of y, the third column contains the numerical coefficients of z and the fourth column contains the constant coefficients from the right hand side of each equation.

Expressing the system of linear equations in this much more compact form has several advantages. First, by relieving us of the tedium of copying the literal coefficients from one line to the next, it allows us to find solutions much more quickly. Second, it allows us to focus entirely on the essential details, which frees us from the pitfall of wasting time on irrelevant information. Finally, this approach lends itself much more easily and neatly to automation (i.e. using a computer or electronic calculator to solve a linear system).

Elementary Row Operations and how they Correspond to the Method of Elimination

There are operations that can be performed on the rows of an augmented matrix that correspond exactly to the steps performed when using the system of elimination. These operations are summarized in the following table.

Elementary Row Operations	*Corresponding Operations in Method of Elimination*
1. Any row can be multiplied or divided by a non-zero constant.	**1.** Both sides of an equation can be multiplied or divided by a non-zero constant.
2. Any row can be replaced by the sum or difference of that row and a multiple of another row.	**2.** A multiple of an equation can be added to or subtracted from any other equation.
3. Any two rows can be interchanged (swapped).	**3.** Two equations can be interchanged (swapped).

Solution to Example from Previous Page using Elementary Row Operations

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & 4 & -7 \\ 3 & -14 & 1 & -48 \end{array}\right)$$

$$\begin{array}{r} \\ -2R_1 + R_2 \rightarrow \\ -3R_1 + R_3 \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -2 & -15 \\ 0 & -20 & -8 & -60 \end{array}\right)$$

$$\begin{array}{r} \\ \\ 4R_2 - R_3 \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -2 & -15 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The final row of this matrix corresponds to the equation $0z = 0$, which of course, has an infinite number of solutions. ***The rest of the solution is shown on the right hand side of this table.***

Solution to Example from Previous Page using Standard Approach

$$\begin{aligned} x + 2y + 3z &= 4 \quad (1) \\ 2x - y + 4z &= -7 \quad (2) \\ 3x - 14y + z &= -48 \quad (3) \end{aligned}$$

$(1) \times -2 + (2),\qquad -5y - 2z = -15 \quad (4)$

$(1) \times -3 + (3),\qquad -20y - 8z = -60 \quad (5)$

$(4) \times 4 - (5),\qquad 0z = 0 \quad (6)$

Since the equation $0z = 0$ has an infinite number of solutions, we can let $z = t$ and write parametric solutions for x and y. By substituting $z = t$ into equation (4), we obtain $y = -\frac{2}{5}t + 3$. By substituting the parametric expressions for y and z into equation (1), we obtain $x = -\frac{11}{5}t - 2$. Summarizing, we conclude that the planes

$$\begin{aligned} x + 2y + 3z &= 4 \quad (1) \\ 2x - y + 4z &= -7 \quad (2) \\ 3x - 14y + z &= -48 \quad (3) \end{aligned}$$

intersect in a line with parametric equations

$$\begin{aligned} x &= -\tfrac{11}{5}t - 2 \\ y &= -\tfrac{2}{5}t + 3 \\ z &= t \end{aligned}$$

Important Terminology

Gaussian Elimination

Gaussian elimination is a matrix method for solving systems of linear equations. It involves the use of elementary row operations to transform a matrix into a form in which the ***entries in the lower triangular portion (below the main diagonal) are all zeros***. A matrix in this form is said to be in ***row-echelon form***. Once the matrix is in row-echelon form, ***back substitution*** needs to be performed to calculate the required values. In the example below, for instance, the third row of the row-echelon matrix corresponds to the equation $p_3 z = l_3$, which produces the solution $z = \dfrac{l_3}{p_3}$. To calculate *y*, this value of *z* must be substituted into the equation corresponding to the second row. Finally, to calculate *x*, the values of *y* and *z* are substituted into the equation corresponding to the first row.

$$\left(\begin{array}{ccc|c} a_1 & a_2 & a_3 & k_1 \\ b_1 & b_2 & b_3 & k_2 \\ c_1 & c_2 & c_3 & k_3 \end{array}\right)$$

Before applying Gaussian elimination

$$\left(\begin{array}{ccc|c} m_1 & m_2 & m_3 & l_1 \\ 0 & n_2 & n_3 & l_2 \\ 0 & 0 & p_3 & l_3 \end{array}\right)$$

After applying Gaussian elimination

Gauss-Jordan Elimination

Gauss-Jordan elimination is a variation of Gaussian elimination. In Gauss-Jordan elimination, elementary row operations are applied until ***all entries are zero except those in the main diagonal, which are all one***. A matrix in this form is said to be in ***reduced row-echelon form***. The main advantage of Gauss-Jordan elimination over Gaussian elimination is that back substitution is not required.

$$\left(\begin{array}{ccc|c} a_1 & a_2 & a_3 & k_1 \\ b_1 & b_2 & b_3 & k_2 \\ c_1 & c_2 & c_3 & k_3 \end{array}\right)$$

Before applying Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & l_3 \end{array}\right)$$

After applying Gauss-Jordan elimination

Example

Use Gauss-Jordan elimination to solve the given system of three linear equations in three unknowns. Before racing ahead and burying your head in the world of elementary row operations, pause for a moment to interpret the given system as the intersection of three planes in $\mathbb{R}^3$. Before you perform even a single elementary row operation, you should know whether the system represents type I, II, III or IV intersection.

$$\begin{aligned} x + y + 2z &= 8 \quad (1) \\ -x - 2y + 3z &= 1 \quad (2) \\ 3x - 7y + 4z &= 10 \quad (3) \end{aligned}$$

Solution

There is no pair of parallel normals, so we can proceed directly to calculating the triple scalar product.

$$\begin{aligned} & n_1 \times n_2 \cdot n_3 \\ &= (1,1,2) \times (-1,-2,3) \cdot (3,-7,4) \\ &= (7,-5,-1) \cdot (3,-7,4) \\ &= 52 \\ &\neq 0 \end{aligned}$$

Therefore, the normal vectors of the three planes are non-coplanar, which means that we should expect a unique solution (type IV, planes intersect at a single point).

Augmented Matrix for the System $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right)$

$$\begin{array}{r} \\ R_2 + R_1 \rightarrow \\ R_3 - 3R_1 \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right)$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ 10R_2 - R_3 \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & 52 & 104 \end{array}\right)$$

$$\begin{array}{r} \\ \\ R_3 \div 52 \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{array}{r} R_1 - 7R_3 \rightarrow \\ R_2 - 5R_3 \rightarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\begin{array}{r} \\ -R_2 \rightarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

This final matrix is in reduced row-echelon form. Clearly, it corresponds to the equations

$$\begin{aligned} x &= 3 \\ y &= 1 \\ z &= 2 \end{aligned}$$

By substituting these values into each of the original equations, we find that each equation is satisfied.

Homework

p. 301: #6, 7, 8, 9 **p. 309:** #7, 8, 9