UNIT 4 – DISCRETE MATHEMATICS AND ITS APPLICATIONS

Counting Techniques

Introduction – Don't we already know how to count?

Didn't Count von Count on Sesame Street teach us everything we need to know about counting? As much as we appreciate the count's efforts, he only taught us ***enumeration***, the most elementary counting principle. This method involves listing objects one by one until the number of items is determined. While this method works extremely well for a small number of items, it is wholly inadequate when dealing with larger sets. Luckily, mathematicians have developed many clever methods that allow us to count in a much more intelligent and efficient manner.

Introductory Problem

The Lotto 6/49® lottery involves selecting six numbers from the set of integers ranging from 1 to 49 inclusive. To win the Lotto 6/49® jackpot, a contestant must match all six of the drawn numbers. What is the probability of doing so?

Analysis of Problem

Although we are not yet in a position to solve the Lotto 6/49® jackpot problem, at this juncture we can discuss why Count von Count's method ***will not work!*** To solve this problem, we need to count the number of ways we can select six (different) integers from the set of integers ranging from 1 to 49 inclusive. Once we learn about ***combinations***, we shall discover that there are 13983816 different ways of choosing a set of six integers from 49. Assuming that you could generate each set at a rate of one per second and never need to take any breaks, it would take almost 162 days to list all the possibilities!

Basic Set Theory and the Additive Counting Principle

Sets

A ***set*** is any collection of objects. The members of a set are called its ***elements***. Each element of a set must be unique, that is, a set may contain only ***one*** "copy" of any given element. The symbol "$\in$" is used to denote membership of a set. For example, "$2 \in A$," read "2 is an element of *A*," means that the number 2 belongs to the set *A*.

Rules for Determining Set Membership

We use ***rules*** to determine whether an element belongs to a set. Very often, these rules can be expressed in terms of formulas. For example, the set $E = \{x \in \mathbb{Z} \mid x = 2k \text{ and } k \in \mathbb{Z}\}$ is the ***set of all even integers***. The formula "$x = 2k$ and $k \in \mathbb{Z}$" determines that the elements of *E* must be even. Furthermore, since this formula can be used to generate ***any even integer***, *E* ***must be*** the set of ***all*** even integers.

$$E = \{x \in \mathbb{Z} \mid x = 2k \text{ and } k \in \mathbb{Z}\}$$

This is read "the set of all integers *x* such that *x* is the product of any integer and two."

Can Rules always be expressed as Formulas?

Mathematical formulas serve as a way of summarizing mathematical relationships in a neat, succinct equation. Sometimes, however, we either do not know how to express a particular rule in terms of a formula or it is not even possible to do so! For instance, consider the set $P = \{p \in \mathbb{Z} \mid p \text{ is prime}\}$, which is the set of all prime numbers.

Unfortunately, nobody has been able to find a formula that generates all the prime numbers. There are ***algorithms*** that in theory can generate all the primes but they are excruciatingly slow. Using such algorithms, even the fastest supercomputers would take ***millions of years*** to generate all the prime numbers less than *N*, where *N* is a sufficiently large natural number.

For instance, consider the problem of finding all 512-bit prime numbers, that is, all prime numbers that can be represented using 512 or fewer binary digits. There are approximately 10^{151} such primes. By contrast, there are only 10^{77} atoms in the universe. If each atom in the universe could generate one billion new primes each ***microsecond***, then from the ***beginning of time until now***, only 10^{109} primes could be generated! To complicate matters, if one could find a way of building a device that could store 1 GB of data per gram of its mass, the device would become so massive that it would collapse upon itself into a black hole! For this reason, prime numbers are of great importance in the field of ***cryptography*** (data encryption).

The Cardinality of Sets

The ***cardinality*** of a set is a measure of its size. The cardinality of a ***finite*** set is simply equal to the number of elements in the set. The cardinality of an infinite set depends on whether the set is ***countable*** or ***uncountable***. An infinite set A is ***countable*** if there exists a one-one correspondence between the elements of A and the set of natural numbers (i.e. $\mathbb{N}$). The infinite set A is ***uncountable*** if there ***does not*** exist a one-one correspondence between the elements of A and those of $\mathbb{N}$. In simple terms, we can think of countable sets as those whose elements can be "numbered" using the natural numbers.

If A is a set, then the cardinality of A is denoted $n(A)$. There are three possibilities for the value of $n(A)$.

1. If A is finite, then $n(A) = k$, where $k \in \mathbb{N}$.
2. If A is countable, then $n(A)$ is said to be **"aleph-null"** or **"aleph-naught."** (This is the cardinality of $\mathbb{N}$.)
3. If A is uncountable, then $n(A)$ is said to be **"aleph-one."** (This is the cardinality of $\mathbb{R}$.)

Discrete Mathematics and Combinatorics

"Discrete mathematics" is the branch of mathematics dealing with problems involving finite or countable sets. Problems involving uncountable sets are generally in the realm of ***calculus***, which in more advanced circles is known as ***analysis***. ***Algebra*** also deals with problems involving uncountable sets but it usually deals with finite or countable properties of uncountable sets. For example, the vector space $\mathbb{R}^3$ The branch of discrete mathematics dealing with counting problems is called ***combinatorics***.

Examples

Finite Sets

$$A = \{n \in \mathbb{N} \,|\, nk = 100, k \in \mathbb{N}\}$$
$$= \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$$
= set of all whole factors of 100

Countable

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$
$$\mathbb{N} = \{1, 2, 3, \ldots\}$$
$$W = \{0, 1, 2, 3, \ldots\}$$
$$P = \{p \in \mathbb{Z} \,|\, p \text{ is prime}\}$$
$$\mathbb{Q} = \left\{q \in \mathbb{R} \,\middle|\, q = \frac{n}{m}, n \in \mathbb{Z}, m \in \mathbb{Z}, m \neq 0\right\}$$

Uncountable

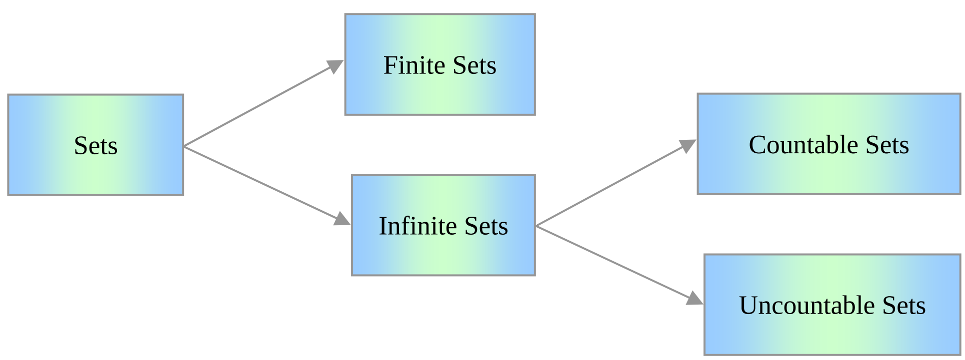
$\mathbb{R}$ =set of real numbers

$$\mathbb{C} = \{a + bi \,|\, a \in \mathbb{R}, b \in \mathbb{R}, i = \sqrt{-1}\}$$

Set of Complex Numbers

Summary

- A set is a collection of objects.
- The symbol "$\in$" is used to denote membership in a set.
- A set can have only one "copy" of each element.
- Sets can be finite, countably infinite or uncountably infinite.
- The cardinality of a set means the number of elements in the set.



Operations on Sets

Just as we can operate on numbers using operations such as addition and multiplication, we can also operate on sets. In this course, we shall discuss the operations union, intersection, complement, subset and cardinality.

The Null (Empty) Set

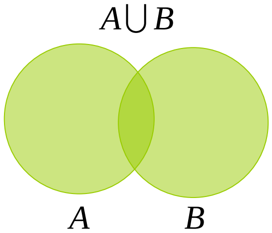
Just as a container can be empty, so can a set! In addition, in the realm of numbers we use the number ***zero*** to indicate a state of nothingness. Therefore, we require some way of representing a set that has no elements. Such a set is called an ***empty*** or ***null set*** and is denoted $\{\ \}$ or ϕ.

Union of two Sets

If A and B are sets, then $A \cup B$, read "the union of A and B," consists of all elements belonging to either A or B. The union of two sets can be thought of as the "joining" of two sets, which results in a larger set. Formally, the union of the sets A and B is defined as follows:

$$A \cup B = \left\{ x \middle| x \in A \text{ or } x \in B \right\}$$

Pictorially, the union of two sets is shown as follows. Such a diagram is called a ***Venn diagram***.

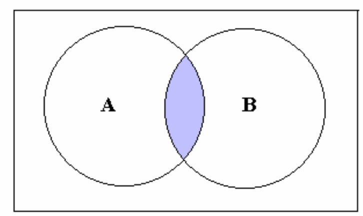


> Notice that since there may be some elements that belong to both A and B, the cardinality of $A \cup B$ is not necessarily equal to the cardinality of A plus the cardinality of B. To calculate $n(A \cup B)$, we need to know how many elements are found in both A and B.

Intersection of two Sets

If A and B are sets, then $A \cap B$, read "the intersection of A and B," consists of all elements belonging to both A ***and*** B. The intersection of two sets can be thought of as a discarding of all elements other than the ones that are common to both sets. This process usually results in a set that is smaller than either A or B. Formally, the intersection of the sets A and B is defined as follows:

$$A \cap B = \left\{ x \middle| x \in A \text{ and } x \in B \right\}$$



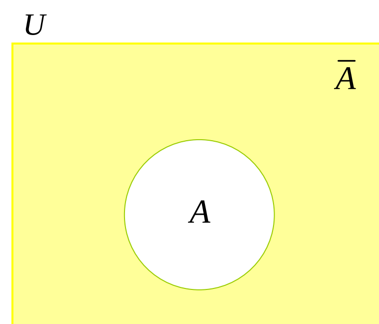
Pictorially, the intersection of two sets is shown as follows. It is the "overlapping" region of the two sets.

Complement of a Set

If A is a set, then its ***complement***, denoted $\overline{A}$, is defined loosely as the set of all elements that are ***not*** elements of A. This concept, however, makes no sense unless we define a set that contains all possible elements of interest. Such a set is called a ***universal set*** and is usually denoted U. Therefore, $\overline{A}$ is defined as

$$\overline{A} = \left\{ x \in U \middle| x \notin A \right\}$$

Pictorially, the complement of A is shown as follows:



Subset of a Set

If A is a set and all its elements are also elements of another set B, then A is called a ***subset*** of B.

Summary

Given a ***universal set*** U and sets A and B consisting of elements found in U,

1. $A \cup B = \left\{ x \in U \middle| x \in A \text{ or } x \in B \right\}$

2. $A \cap B = \left\{ x \in U \middle| x \in A \text{ and } x \in B \right\}$

3. $\overline{A} = \left\{ x \in U \middle| x \notin A \right\}$

4. If every element of A is also an element of B, then we say that A is a ***subset*** of B. In this case, if $A \neq B$ (i.e. A is "smaller than" B) we say that A is a ***proper subset*** of B and we write $A \subset B$. If we know that A is a subset of B but A might be equal to B, then we write $A \subseteq B$. (This is similar to the use of the symbols "$<$" and "$\leq$.")

Examples

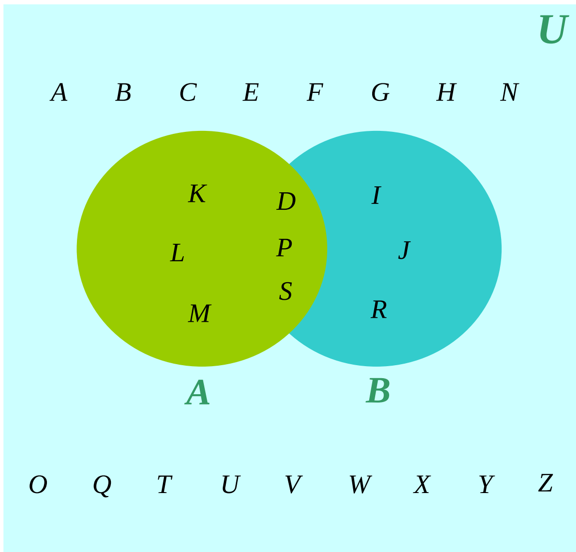
1. Let $U=\{A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z\}$ = set of all letters of the alphabet
   $A=\{D,K,L,M,P,S\}$ = set of first letters of surnames of students in a math class
   $B=\{D,I,J,P,R,S\}$ = set of first letters of given names of students in the same math class

   (a) $n(U)=26$, $n(A)=6$, $n(B)=6$

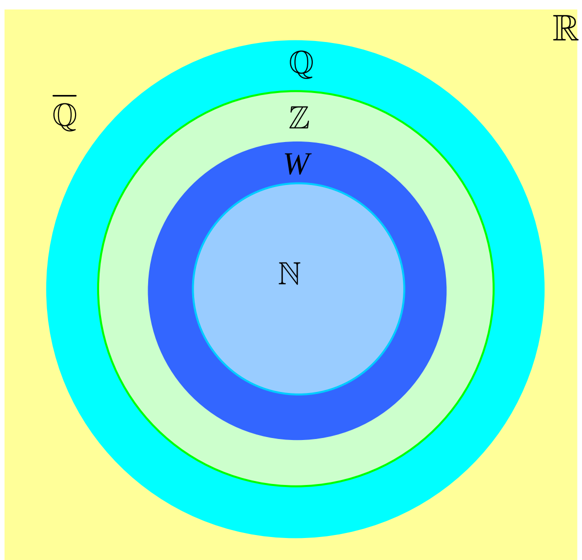
   (b) $A\cup B=\{D,I,J,K,L,M,P,R,S\}$, $A\cap B=\{D,P,S\}$,
   $\overline{A}=\{A,B,C,E,F,G,H,I,J,N,O,Q,R,T,U,V,W,X,Y,Z\}$

   (c) $n(A\cup B)=9\neq n(A)+n(B)=6+6=12$

   (d) $n(A\cup B)=9=6+6-3=n(A)+n(B)-n(A\cap B)$



2. Create a Venn diagram that shows the relationship among the sets $\mathbb{N}$, $\mathbb{Z}$, W, $\mathbb{Q}$, $\overline{\mathbb{Q}}$ and $\mathbb{R}$. For the purposes of this question, use $\mathbb{R}$ as the universal set.



The Additive Counting Principle

As we observed earlier, the cardinality of $A \cup B$ ***is not*** equal to the cardinality of A plus the cardinality of B. When we add $n(A)$ and $n(B)$, we are counting ***twice*** the elements that A and B have in common. In mathematical terms, we can write this as follows:

$$n(A)+n(B)=n(A\cup B)+n(A\cap B)$$

Therefore,

$$n(A\cup B)=n(A)+n(B)-n(A\cap B)$$

Summary

If A and B are finite subsets of the finite universal set U, then

1. $n(A\cup B)=n(A)+n(B)-n(A\cap B)$
2. $n(A\cup B)=n(A)+n(B)$ ***if and only if*** $A\cap B=\{\ \}=\phi$ (i.e. iff A and B have no elements in common)
3. If the intersection of two or more sets is the empty set, we say that the sets are ***disjoint***. This means that the previous statement can be rephrased as follows:

   $n(A\cup B)=n(A)+n(B)$ ***if and only if*** A and B are disjoint
4. By applying property 3 to the disjoint A sets and $\overline{A}$, we can immediately derive the following ***corollary***: $n(U)=n(A\cup\overline{A})=n(A)+n(\overline{A})$
5. By rearranging property 4, we immediately see that $n(A)=n(U)-n(\overline{A})$.

The Multiplicative Counting Principle

Example

Mr. Nolfi is concerned about three students, C. H. Eat, S. C. Ribe and I. C. Opy, who have a tendency to cheat on tests. To minimize the probability of cheating, he has decided to use a somewhat bizarre but effective strategy. Every ten minutes, the students will be forced to move to a different desk. Since the three desks are side-by-side, all Mr. Nolfi has to do is work out all the possible arrangements of the three students. The diagram to the right should help you understand this idea.

Time (min)	Desk 1	Desk 2	Desk 3
0-10	C. H. Eat	S. C. Ribe	I. C. Opy
10-20	C. H. Eat	I. C. Opy	S. C. Ribe
20-30	S. C. Ribe	I. C. Opy	C. H. Eat
30-40	S. C. Ribe	C. H. Eat	I. C. Opy
40-50	I. C. Opy	C. H. Eat	S. C. Ribe
50-60	I. C. Opy	S. C. Ribe	C. H. Eat

Notice that if we arrange the students from left-to-right, there are ***three*** choices for the student who sits at the first desk, ***two*** choices for the student who sits at the second desk and ***one*** choice for the student who sits at the third desk. Therefore, for ***each student*** who sits at the first desk, there are ***two*** choices for the student who sits at the second desk. We can count the total number of seating arrangements then, simply by multiplying 3 by 2 because we have ***three groups of two arrangements*** when we fix the student who sits at the first desk. This example helps us to understand ***the multiplicative counting principle***. It will also lead us to the notion of the ***factorial*** function.

The Factorial Function

1. The symbol $n!$ (read "n factorial") represents the product of all consecutive integers from 1 to n inclusive. That is, $n!=n(n-1)(n-2)\cdots(2)(1)$ for all $n\in\mathbb{N}$. In addition, for the sake of convenience when writing formulas involving factorials, 0! is ***defined to be*** 1.

2. The factorial function can also be defined ***recursively*** as shown below.

   For all $n\in\mathbb{N}\cup\{0\}$, $n!=\begin{cases}0, & \text{if } n=0\\ n(n-1)!, & \text{if } n\geq 1\end{cases}$

Product Rules (Multiplicative Principles)

1. **_Product Rule (Multiplicative Principle)_**
   Let t_1, t_2 represent a sequence of 2 tasks. If t_1 can be performed in m_1 ways and for each of these, t_2 can be performed in m_2 ways, then the sequence of two tasks can be performed in $m_1 m_2$ ways altogether.
2. **_Generalized Product Rule (Generalized Multiplicative Principle)_**
   Let $t_1, t_2, \ldots, t_n$ represent a sequence of n tasks. Suppose that task t_1 can be performed in m_1 ways and for each of these ways, task t_2 can be performed in m_2 ways, and for each of these ways, task t_3 can be performed in m_3 ways, and so on. Then, the sequence of tasks can be performed in $m_1 m_2 \cdots m_n = \prod_{i=1}^{n} m_i$ ways altogether.

   This symbol is similar to sigma notation except that it is used for products instead of sums. The uppercase Greek letter pi (Π) is used to stand for "product" because it is the Greek equivalent of the Latin letter "P."
3. **_Corollary of Generalized Product Rule_**
   If task t_1 can be performed in 1 way and for each of these ways, task t_2 can be performed in 2 ways, … and for each of these ways, task t_n can be performed in n ways, then the sequence of tasks can be performed in $n! = n(n-1)(n-2)\cdots(2)(1)$ ways.

Example 1

You own 3 shirts, 2 pairs of pants and 2 pairs of socks. Assuming that you ***always*** put on your shirt first, your pants second and your socks third, in how many ways can you dress yourself?

Solution

Let S_1, S_2, S_3 represent the three shirts, p_1, p_2 represent the pants and s_1, s_2 represent the socks. Then, the set given below contains all the possible arrangements:

$$\{S_1p_1s_1, S_1p_1s_2, S_1p_2s_1, S_1p_2s_2, S_2p_1s_1, S_2p_1s_2, S_2p_2s_1, S_2p_2s_2, S_3p_1s_1, S_3p_1s_2, S_3p_2s_1, S_3p_2s_2\}$$

By counting the number of elements in this set, we see that there are 12 possible arrangements. By using the generalized product rule, however, all we need to do is the following:

$$\text{number of ways} = 3 \times 2 \times 2 = 12$$

Example 2

A video card can display graphics in a variety of different modes including 64-bit colour at a resolution of 1600 pixels × 1200 pixels.

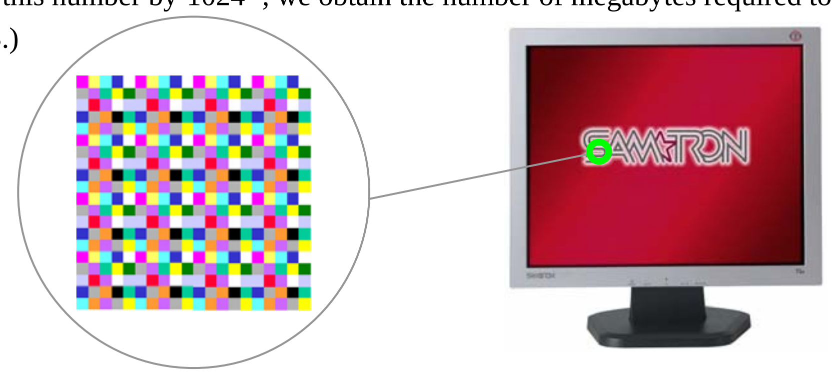
Solution

(a) Each colour is represented as a sequence of 64 bits ("binary digits") such as 1101100100100100010110010010010001011001001001000100100100100101. Using the generalized multiplicative principle, the number of such sequences is equal to

$$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{64\text{ times}} = 2^{64} \doteq 1.844674407 \times 10^{19}.$$

(b) One byte of memory is equal to 8 bits. Since each colour requires 64 bits, the number of bytes required to store the colour code of one pixel is $64 \div 8 = 8$. Therefore, each pixel requires 8 bytes of storage (memory). The number of pixels in such an image is equal to $1600 \times 1200 = 1920000$. Therefore, the number of bytes required to store the image $= 1920000 \times 8 = 15360000$ bytes. (By dividing this number by 1024^2, we obtain the number of megabytes required to store the image: $15360000 \div 1024^2 \doteq 14.65$ MB.)

This is a ***magnified*** section of an image on a computer monitor screen. Each small square is called a ***pixel*** ("picture element"), which is the smallest addressable segment of the picture. The colour of each pixel is determined by a binary code.



Homework for Sets, Additive Counting Principle, Multiplicative Counting Principle

Sets and Subsets	*Sum Rules*	*Product Rules*	*Miscellaneous*
pp.359 - 361 #5, 6, 8, 9, 11, 13, 14, 15 **pp.364 - 366** #4, 5, 6, 8, 9, 12, 13, 14, 15	**pp.370 - 371** #5, 7, 8, 10, 12, 14, 15, 16	**pp.377 - 380** #4, 6, 7, 9, 12, 13, 14, 15, 16	**pp.384 - 385** #2, 4, 8, 9, 11, 13, 15, 16

Sequences and Subsets

- ***Sequence***

  An ***ordered*** set of quantities (called "terms").

  Example 1: 1, 2, 4, 8, 16, 32, 64.
  Example 2: 64, 32, 16, 8, 4, 2, 1

  Notice that although each sequence contains the same integers, the sequences are ***not the same*** because the integers are listed in a different order.

  For sequences, order matters!

- ***Subset***

  A set whose members are members of another set

  Example: $\{1,2,4,8,16,32,64\}$ is a subset of $\{1,2,4,8,16,32,64,128,256\}$

  For sets, order does not matter! All that matters is membership. A particular value either is an element of a set or it is not an element of a set. For instance, the set $A=\{1,2,4,8,16,32,64\}$ is equal to the set $B=\{64,32,16,8,4,2,1\}$ because their members are identical. (In more precise terms, $A=B$ if and only if $A\subseteq B$ and $B\subseteq A$.)

From this discussion, the following is clear: ("↔" means "is equivalent to")

Counting Groupings when Order Matters ↔ Counting Number of Sequences
Counting Groupings when Order Doesn't Matter ↔ Counting Number of Subsets

Counting Sequences with Distinct Terms (Groupings with Distinct Objects when Order Matters)

Example

An ***anagram*** is a word or phrase spelled by rearranging the letters of another word or phrase (each letter can be used only once in the rearrangement). How many six-letter anagrams of the word "sycophant" are there? (For the purposes of this question, it does not matter whether the anagram that is formed is a valid English word.)

Solution

The easiest approach to solving this type of problem is once again, to ***draw a picture!*** Imagine that you have six spaces available waiting to be filled with the letters found in the word "sycophant." Then fill the spaces by freely choosing letters. There are nine choices for the leftmost letter of the word. Once the leftmost letter is chosen, there are eight choices for the next letter. Continuing in this manner, we have 7, 6, 5 and 4 choices respectively for the remaining letters. By applying the ***multiplicative counting principle***, the number of anagrams must be equal to

$$9\times 8\times 7\times 6\times 5\times 4=60480.$$

$$\underline{9}\times\underline{8}\times\underline{7}\times\underline{6}\times\underline{5}\times\underline{4}$$

Number of Choices for 1st Letter	Number of Choices for 2nd Letter	Number of Choices for 3rd Letter	Number of Choices for 4th Letter	Number of Choices for 5th Letter	Number of Choices for 6th Letter

Calculations like the one done above can be extremely tedious. By observing the following, however, we can reduce dramatically the amount of work required:

$$9\times 8\times 7\times 6\times 5\times 4=\frac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{3\times 2\times 1}=\frac{9!}{3!}$$

Permutations

The number of sequences of length r that can be formed using n different symbols if each symbol can be used only once is given by $P(n,r)={}_nP_r=n(n-1)\cdots(n-r+1)=\dfrac{n!}{(n-r)!}$. Each sequence that is formed from the symbols is called a ***permutation*** of the symbols. (Note that the symbols $P(n,r)$ and ${}_nP_r$ are interchangeable.)

Counting Sequences with Unlimited Repeating Terms (Groupings with Unlimited Repeating Objects when Order Matters)

Example 1

Suppose that you were allowed to use each letter in a word ***as many times as you like*** when forming an ***anagram***. How many six-letter anagrams of the word "sycophant" are there if you can use each letter as many times as you like? (For the purposes of this question, it does not matter whether the anagram that is formed is a valid English word.)

Solution

The easiest approach to solving this type of problem is once again, to ***draw a picture!*** Imagine that you have six spaces available waiting to be filled with the letters found in the word "sycophant." Then fill the spaces by freely choosing letters. Since you are allowed to use each letter an unlimited number of times, there are ***nine choices for each space that needs to be filled***. By applying the ***multiplicative counting principle***, the number of anagrams (with unlimited repetitions) must be equal to

$$9\times9\times9\times9\times9\times9=9^6=531441.$$

$$\underline{9}\times\underline{9}\times\underline{9}\times\underline{9}\times\underline{9}\times\underline{9}$$

Number of Choices for 1st Letter	Number of Choices for 2nd Letter	Number of Choices for 3rd Letter	Number of Choices for 4th Letter	Number of Choices for 5th Letter	Number of Choices for 6th Letter

Sequences with Unlimited Repeated Terms

The number of sequences of length r that can be formed using n different symbols if each symbol can be used as often as we like is $\underbrace{n(n)\cdots(n)}_{r\text{ factors of }n}=n^r$.

Example 2

Consider all 32-bit colour codes (sequences of binary digits of length 32).

(a) How many different 32-bit codes are there altogether?

(b) How many of the 32-bit sequences have ***at least one*** "0" bit?

(c) How many of the codes begin with a "0" and end with a "1?"

Solution

(a) Let U represent the set of all 32-bit sequences. Since there are only two different symbols (0 and 1) we can use the principle stated above to conclude that $n(U)=2^{32}$.

(b) Let A represent the set of all elements of U that have at least one "0" bit. Then $\overline{A}$ must represent the set of all elements of U that have ***no "0" bits!*** Obviously, there is only one string of 32 bits that does not have any zeros (11111111111111111111111111111111). Therefore,

$$\begin{aligned}n(A)&=n(U)-n(\overline{A})\\&=2^{32}-1\end{aligned}$$

(c) Let B represent the set of all elements of U that have at begin with a "0" bit and end with a "1" bit. Since the first and last bits of each sequence are fixed, $n(B)$ is equal to the number of sequences that can be formed using the 30 bits that lie between the first and last bits. Therefore,

$$n(B)=2^{30}.$$

Homework for Counting Sequences

pp.400 - 401	**pp.404 - 407**
#5, 6, 7, 9, 11, 12, 15, 16, 18	#5, 6, 7, 10, 11, 12, 13, 15, 18, 21, 24

Counting Subsets (Groupings when Order doesn't Matter)

Back to the Lotto 6/49 Problem

We are finally in a position to solve the Lotto 6/49 problem stated at the beginning of this unit. Before we consider a solution, however, we should first rephrase the problem using the mathematically precise language of sequences and subsets. Using this language, the Lotto 6/49 problem can be stated as follows:

Let $L = \{1, 2, 3, \ldots, 47, 48, 49\} = \{n \in \mathbb{N} \mid 1 \le n \le 49\}$ and $S = \{A \subset L \mid n(A) = 6\}$. What is $n(S)$?

In less precise but perhaps more intuitive language, we can see that the Lotto 6/49 problem reduces to counting the number of subsets of size six that can be formed using the natural number numbers from 1 to 49 inclusive.

This problem can be solved easily if we break it up into two parts. First we shall determine how many sequences with six distinct terms can be formed using the integers from 1 to 49 inclusive. The number of sequences, however, is much greater than the correct answer to this problem because for sequences, order matters, while for subsets, order does not matter. Therefore, the second step is to eliminate all the ***permutations*** (rearrangements) of a given group of numbers.

1. ***How many sequences with six distinct terms can be formed using the integers from 1 to 49 inclusive?***
   From the considerations of a previous section, the answer is obviously $P(49,6)$.

2. ***Given a set whose cardinality is six, how many permutations can be formed using all six elements of the set?***
   Again from the considerations of a previous section, the answer is obviously 6!. (For example, there are 6! ways of rearranging the integers 1, 2, 3, 4, 5 and 6. However, all 6! arrangements represent the same Lotto 6/49 ticket.)

Therefore, the number of different tickets that can be played in Lotto 6/49 is equal to

$$\frac{P(49,6)}{6!} = \frac{49!}{43!} \div 6! = \frac{49!}{43!} \times \frac{1}{6!} = \frac{49!}{6!43!} = 13983816$$

Combinations (Number of Subsets)

The number of subsets *S* of size *r* (i.e. $n(S) = r$) of a set *A* of size *n* (i.e. $n(A) = n$)is given by

$$C(n,r) = {}_nC_r = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Each subset of size *r* is called a ***combination*** of the symbols of the set *A*. The symbols $C(n,r)$, ${}_nC_r$ and $\binom{n}{r}$ are interchangeable and are all read "*n* choose *r*." In mathematical literature $\binom{n}{r}$ is used most commonly, a practice to which we shall adhere in this course. On calculators, however, you are far more likely to find the symbols $C(n,r)$ and ${}_nC_r$.

Example 1

How many different poker hands are possible?

Solution

There are 52 cards in a standard deck and 5 cards in a poker hand. Therefore, the total number of poker hands possible is equal to the number of ways of choosing 5 cards from a deck of 52. In other words, the number of hands is equal to the number of subsets of size 5 of a set of size 52, or "52 choose 5."

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{\overset{13}{\cancel{52}} \times \overset{17}{\cancel{51}} \times \overset{10}{\cancel{50}} \times 49 \times \overset{24}{\cancel{48}} \times \cancel{47!}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{47!}} = \frac{13 \times 17 \times 10 \times 49 \times 24 \times 1}{1 \times 1 \times 1 \times 1 \times 1 \times 1} = 2598960$$

Of course, if you have a scientific or graphing calculator, there is no need to complete the intermediate steps shown above. However, it is instructive to see how the recursive definition of the factorial function ($n! = n(n-1)!$) can be exploited to simplify the arithmetic involved in computing permutations and combinations.

Investigation

1. Evaluate each of the following.

$\binom{9}{0}$	$\binom{9}{9}$
$\binom{9}{1}$	$\binom{9}{8}$
$\binom{9}{2}$	$\binom{9}{7}$
$\binom{9}{3}$	$\binom{9}{6}$
$\binom{9}{4}$	$\binom{9}{5}$

2. Now study your results carefully. What do you notice? Can you explain what you observe? Can this result be generalized? If so, state a conjecture that applies to $\binom{n}{r}$ for all values of n and r.

Example 2

UTS (University of Toronto Schools), a private school affiliated with the University of Toronto, offers grades seven to twelve inclusive. Part I of the entrance examination is written by several hundred grade six students, from which the top 100 boys and 100 girls are chosen to write Part II. Each year, 110 of the 200 students who write Part II of the entrance exam are admitted to UTS.

(a) In how many ways can 110 candidates be chosen from the 200 students who write Part II?

(b) UTS does not accept the top 110 students who write Part II regardless of gender. Instead, they select the top 55 girls and 55 boys who write Part II. Given this restriction, in how many ways can the 110 candidates be chosen?

Solution

Before you begin, you should ask yourself an important question. Which answer should be greater, that for (a) or (b)? Since (b) is restricted and (a) is not, the answer to (a) should be greater.

(a) Obviously, the number of ways in which the candidates are chosen is given by $\binom{200}{110}=\dfrac{200!}{90!100!}$, which is too large to be evaluated by most scientific calculators. However, using a graphing calculator we find that $\binom{200}{110} \doteq 3.342221319\times10^{58}$

(b) There are $\binom{100}{55}$ ways of choosing the boys and $\binom{100}{55}$ ways of choosing the girls. Using the multiplicative counting principle, the total number of ways is equal to $\binom{100}{55}\binom{100}{55}=\binom{100}{55}^2 \doteq 3.775914615\times10^{57}$. As we predicted, the answer for (a) is greater than that for (b).

Homework

pp. 413-415 #6, 7, 8, 11, 18

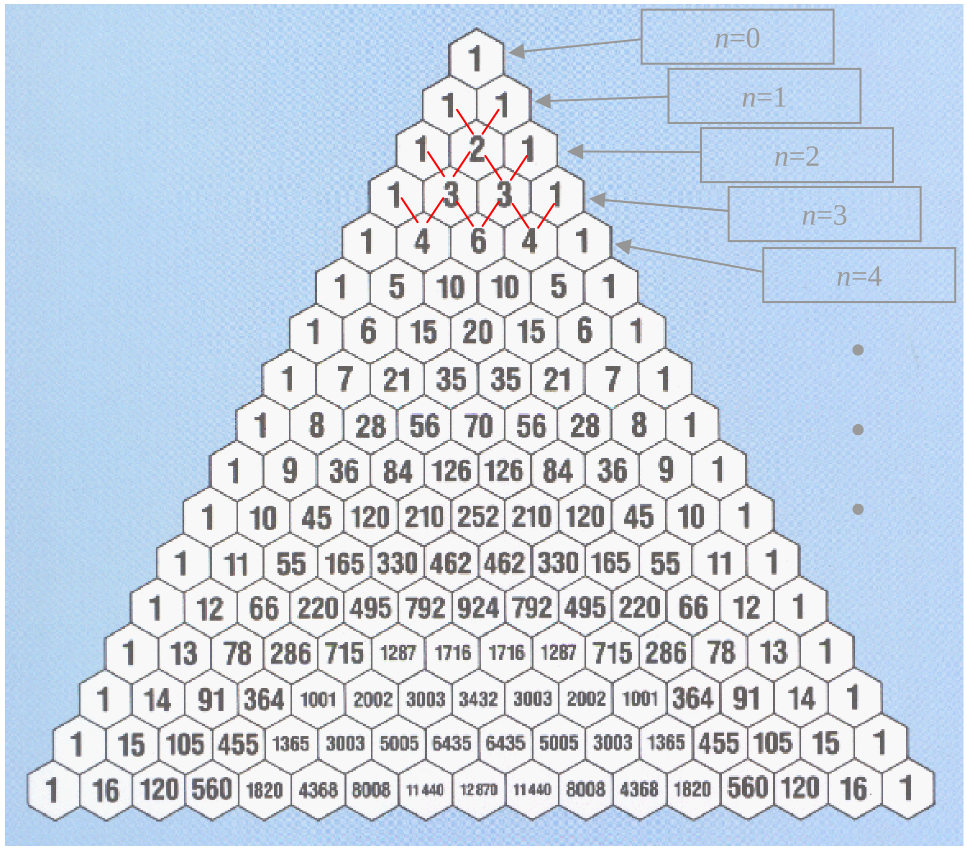
Investigation: Pascal's Triangle

Blaise Pascal (June 19, 1623 – August 19, 1662) was a French mathematician, physicist and religious philosopher. Important contributions by Pascal to the natural sciences include the construction of mechanical calculators, considerations on probability theory, the study of fluids, and clarification of concepts such as pressure and vacuum.

Among high school students, however, Pascal is much more famous for his triangle (shown below). Although the triangle can be generated very easily using simple arithmetic, it is intimately related both combinations and expansions of binomials. ***The purpose of this investigation is to discover how Pascal's triangle is related to combinations and binomial expansions.***

Activity

1. Explain how Pascal's triangle is generated. (The "little red line segments" are meant as a hint.)



2. Now you will delve into the relationship between Pascal's triangle and combinations.

Complete the following table and then compare your results to Pascal's triangle. What do you notice? Do you think that it is true in general?

$n = 0$ $\quad \binom{0}{0} =$

$n = 1$ $\quad \binom{1}{0} = \quad \binom{1}{1} =$

$n = 2$ $\quad \binom{2}{0} = \quad \binom{2}{1} = \quad \binom{2}{2} =$

$n = 3$ $\quad \binom{3}{0} = \quad \binom{3}{1} = \quad \binom{3}{2} = \quad \binom{3}{3} =$

$n = 4$ $\quad \binom{4}{0} = \quad \binom{4}{1} = \quad \binom{4}{2} = \quad \binom{4}{3} = \quad \binom{4}{4} =$

$n = 5$ $\quad \binom{5}{0} = \quad \binom{5}{1} = \quad \binom{5}{2} = \quad \binom{5}{3} = \quad \binom{5}{4} = \quad \binom{5}{5} =$

3. Now expand and simplify each of the following binomials. Once you are finished, compare the numerical coefficients of the expanded polynomials to the rows in Pascal's triangle. What do you notice? State a conjecture based on your results.

$(a+b)^0$

$(a+b)^1$

$(a+b)^2$

$(a+b)^3$

$(a+b)^4$

$(a+b)^5$

Conclusions

Based on the investigation that you have just completed, state two conjectures regarding the relationship among Pascal's triangle, combinations and binomial expansions.

1.

2.

THE BINOMIAL THEOREM

The Connection among Pascal's Triangle, Combinations and the Binomial Theorem

If you were sufficiently observant while conducting the investigation on the previous two pages, you would have probably made the following ***conjectures***:

1. Each row (precisely, row n) of Pascal's triangle takes the form

$$\binom{n}{0} \qquad \binom{n}{1} \qquad \binom{n}{2} \qquad \cdots \qquad \binom{n}{n-1} \qquad \binom{n}{n}$$

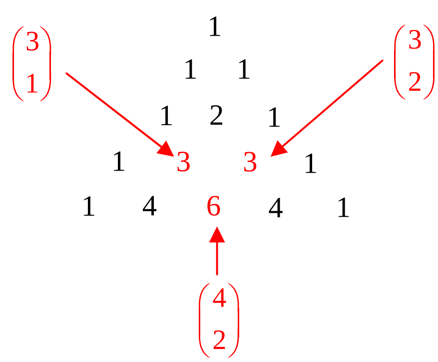
2. The numerical coefficients of the expansion of $(a+b)^n$ are equal to the numbers found in row n of Pascal's triangle. For instance, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$. The coefficients 1, 3, 3 and 1 can be read directly from row 3 of Pascal's triangle.

Therefore, it is reasonable to conjecture that

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \\ &= \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}a^1b^{n-1} + \binom{n}{n}a^0b^n \\ &= \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i\end{aligned}$$

Now remember that at this point, we cannot yet assert that the above statements are true. ***We can only claim that in the examples that we have investigated, the above statements hold true.*** A little more consideration is required to elevate the above statements to the status of "true."

Pascal's Identity



If you recall how Pascal's triangle is formed, and accept that statement 1 above is true, then you must also accept that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

In fact, statement 1 is true if and only if $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

To understand why this should be the case, examine the diagram at the right. Any value in row n+1 is obtained by adding the two values directly "above" it in row n. For example, we know that $\binom{4}{2} = 6$ is the "middle" value in row 4 of Pascal's triangle. We also know that this value is obtained by adding the "3's" directly above it. Stated more precisely, we see that $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$ since $\binom{3}{1} = 3$ and $\binom{3}{2} = 3$. Since we can perform exactly the same steps anywhere in Pascal's triangle, it seems very reasonable to expect that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Theorem: Pascal's Identity

For all $n \in W$ and $k \in \mathbb{N}$ such that $1 \le k \le n$, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Proof:

$$
\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{kn!}{k(k-1)!(n-k+1)!} + \frac{(n-k+1)n!}{k!(n-k+1)(n-k)!} \\
&= \frac{kn! + (n-k+1)n!}{k!(n-k)!} \\
&= \frac{n!(k+(n-k+1))}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}
\end{aligned}
$$

Corollary of Pascal's Identity

For all $n \in W$, the entries in row n of Pascal's triangle are $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$.

A Simple Consequence of Pascal's Triangle

$\binom{n}{k} = \binom{n}{n-k}$ (This is easy to see because of the symmetry of Pascal's triangle.)

Proof:

$$
\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(n-k)!k!} \\
&= \frac{n!}{(n-k)!(n-(n-k))!} \\
&= \binom{n}{n-k}
\end{aligned}
$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Notice the symmetry of Pascal's triangle

Finally, we can state the binomial theorem:

The Binomial Theorem

For all $n \in W$, $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$.

We are not yet in a position to prove the binomial theorem. However, since we have provided a strong inductive argument in favour of it, we shall accept the theorem for the time being and focus on some simple applications.

Examples

$$(x+y)^5 = \sum_{k=0}^{5}\binom{5}{k}x^{5-k}y^k$$
$$= x^5 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+1)^7 = \sum_{k=0}^{7}\binom{7}{k}x^{7-k}1^k$$
$$= \sum_{k=0}^{7}\binom{7}{k}x^{7-k}$$
$$= x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$$

$$(2m+3n^2)^4 = \sum_{k=0}^{4}\binom{4}{k}(2m)^{4-k}(3n^2)^k$$
$$= \sum_{k=0}^{4}\binom{4}{k}2^{4-k}3^k m^{4-k}n^{2k}$$
$$= \binom{4}{0}2^4 3^0 m^4 n^0 + \binom{4}{1}2^3 3^1 m^3 n^2 + \binom{4}{2}2^2 3^2 m^2 n^4 + \binom{4}{3}2^1 3^3 m^1 n^6 + \binom{4}{4}2^0 3^4 m^0 n^8$$
$$= 1(16)(1)m^4 + 4(8)(3)m^3n^2 + 6(4)(9)m^2n^4 + 4(2)(27)mn^6 + 1(1)(81)n^8$$
$$= 16m^4 + 96m^3n^2 + 216m^2n^4 + 216mn^6 + 81n^8$$

What is the ***fifteenth term*** in the expansion of $(4a^2+b^3)^{20}$?

$$(4a^2+b^3)^{20} = \sum_{k=0}^{20}\binom{20}{k}(4a^2)^{20-k}(b^3)^k$$
$$= \sum_{k=0}^{20}\binom{20}{k}4^{20-k}a^{40-2k}b^{3k}$$

For the fifteenth term, k = 14 (for the first term, k = 0, for the second term, k = 1, etc.). Therefore, the fifteenth term is

$$\binom{20}{14}4^{20-14}a^{40-28}b^{42} = \frac{20!}{14!6!}4^6a^{12}b^{42} = 158760960a^{12}b^{42} .$$

Homework

pp. 469-473 #1ef, 4a, 8, 9, 15, 29

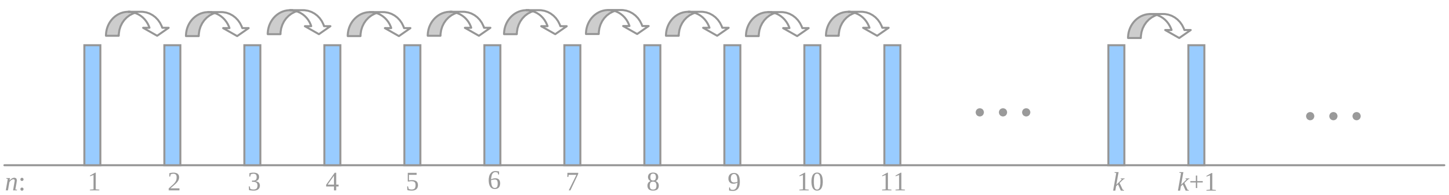
Proof by Mathematical Induction

Introduction

Proof by mathematical induction is a specialized technique of proof that applies specifically to mathematical statements involving the natural numbers. Since there is a direct association between sequences and $\mathbb{N}$, we can equivalently describe mathematical induction as a method of ***proving sequences of mathematical statements***.

Analogy: The Domino Effect

Mathematical induction works because of exactly the same principle that governs the domino effect. Imagine an infinite sequence of dominoes arranged in such a manner that if a particular domino is toppled, then the one immediately following it will also fall over. In such an arrangement, if the first domino in the sequence is toppled, a "chain reaction" is triggered, causing all subsequent dominoes eventually to be knocked down. Consider the following side view of an infinite sequence of dominoes:



Let D_n represent the nth domino in this infinite sequence of dominoes. Then, it is clear that the following properties hold:

1. D_1 is capable of being toppled.

2. For all $n \in \mathbb{N}$, if D_n is toppled, then D_{n+1} will be toppled (i.e. $D_n \to D_{n+1}$).

> ***The Principle of Mathematical Induction***
>
> Let $P_1, P_2, P_3, \ldots, P_k, P_{k+1}, \ldots$ represent an infinite sequence of mathematical statements. In addition, ***suppose that BOTH of the following conditions hold.***
>
> **1.** Statement P_1 is true. (This is known as the ***BASE CASE*** or ***BASAL CASE***. Think of it as the foundational case.)
> AND
> **2.** For all $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. (i.e. $P_k \to P_{k+1}$)
>
> Then P_n must be true for all $n \in \mathbb{N}$.

Examining only a few terms of a sequence of statements will help us to accept this principle. If property 1 holds, then we know that P_1 must be true. If we know P_1 to be true, however, we can apply property 2 to prove that P_2 must be true. Now that we know that P_2 is true, we can apply property 2 again to prove that P_3 must be true. This "chain reaction," just as with the domino effect, continues ad infinitum. This in turn allows us to conclude that every statement in the sequence must be true.

Example

Prove that $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$

Analysis

Before we blindly forge ahead on an inductive excursion, let us take some time to interpret the given statement. The ***left side of the equation*** represents the sum

$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots\binom{n}{n}$

= (# ways of choosing subsets of size 0) + (# ways of choosing subsets of size 1) + (# ways of choosing subsets of size 2) + ⋯ + (# ways of choosing subsets of size n)

= (total # of subsets of a set containing n elements)

The ***right side of the equation*** represents the number of sequences of length n that can be formed using two different symbols if each symbol can be used an unlimited number of times. Specifically, 2^n is equal to the number of different binary sequences of length n. We can use the binary sequences of length n to describe all the subsets of a set containing n elements in the following way. (For the sake of simplicity, the case $n = 4$ is used to illustrate the general case.)

Let $S=\{x_1,x_2,x_3,x_4\}$ represent any set containing 4 elements. In addition, let the binary digit "0" mean "is not an element of S" and let "1" mean "is an element of S." Then the binary sequence 1010, for instance, represents the subset $A=\{x_1,x_3\}$ because the first "1" means that $x_1 \in S$ while the second "1" means that $x_3 \in S$. Similarly, the zeros mean respectively that $x_2 \notin S$ and $x_4 \notin S$.

In this way, we can see that the number of binary sequences of length n must also equal the number of subsets of a set containing n elements. This gives us a great deal of confidence that the statement $\sum_{k=0}^{n}\binom{n}{k}=2^n$ must be true. To settle the question once and for all, we can write a formal proof by mathematical induction as shown below.

Proof

Let P_n represent the statement $\sum_{i=0}^{n}\binom{n}{i}=2^n$.

1. *Base Case*

First we must show that P_0 is true (in this case the induction begins at $n = 0$).

$$\text{L.H.S.}=\sum_{i=0}^{0}\binom{0}{i} =\binom{0}{0} =1 \qquad \text{and} \qquad \text{R.H.S.}=2^0 =1$$

Since L.H.S. = R.H.S. , the base case must be true. Therefore, P_0 must be true.

2. *Inductive Case*

Suppose that the statement P_n is true for $n = k$ (i.e. suppose that P_k is true).

Then, $\sum_{i=0}^{k}\binom{k}{i}=2^k$. (This is called the ***induction hypothesis***.) Now consider $\sum_{i=0}^{k+1}\binom{k+1}{i}$.

The statement P_{k+1} corresponds to $\sum_{i=0}^{k+1}\binom{k+1}{i}=2^{k+1}$. We need to show that this is true if P_k is true.

$$
\begin{aligned}
\sum_{i=0}^{k+1}\binom{k+1}{i} &= \binom{k+1}{0}+\sum_{i=1}^{k}\binom{k+1}{i}+\binom{k+1}{k+1}\\
&=1+\sum_{i=1}^{k}\left[\binom{k}{i}+\binom{k}{i-1}\right]+1\\
&=1+\sum_{i=1}^{k}\binom{k}{i}+\sum_{i=1}^{k}\binom{k}{i-1}+1\\
&=\left[1+\sum_{i=1}^{k}\binom{k}{i}\right]+\left[\sum_{i=1}^{k}\binom{k}{i-1}+1\right]\\
&=\left[\binom{k}{0}+\sum_{i=1}^{k}\binom{k}{i}\right]+\left[\sum_{i=0}^{k-1}\binom{k}{i}+\binom{k}{k}\right]\\
&=\sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k}\binom{k}{i}\\
&=2^k+2^k\\
&=2(2^k)\\
&=2^{k+1}
\end{aligned}
$$

Here we are "splitting off" the first and last terms of the sum.

Here we are applying Pascal's identity.

Here we are applying the rule $\sum(a_i+b_i)=\sum a_i+\sum b_i$, which is simply a more compact form of $(a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n)=(a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n)$

Therefore, we have shown that P_1 is true ***and*** that if P_k is true, then P_{k+1} must also be true. By the principle of mathematical induction, this implies that P_n must be true for all $n \in \mathbb{N}$.

Proof of the Binomial Theorem

Let P_n represent the statement $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$.

1. *Base Case*

First we must show that P_0 is true (in this case the induction begins at $n = 0$).

$$\text{L.H.S.} = \sum_{i=0}^{0} \binom{0}{i} a^{0-i} b^0 \qquad \text{and} \qquad \text{R.H.S.} = (a+b)^0$$

$$= \binom{0}{0} a^0 b^0 \qquad\qquad = 1$$

$$= 1$$

Since L.H.S. = R.H.S. , the base case must be true. Therefore, P_0 must be true.

2. *Inductive Case*

Suppose that the statement P_n is true for $n = k$ (i.e. suppose that P_k is true).

Then, $(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$. (This is called the ***induction hypothesis***.) Now consider $\sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$.

We need to show that this expression is equal to $(a+b)^{k+1}$.

$$\sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i$$

$$= \binom{k+1}{0} a^{k+1} b^0 + \sum_{i=1}^{k} \binom{k+1}{i} + \binom{k+1}{k+1} a^0 b^{k+1}$$

Here we are "splitting off" the first and last terms of the sum.

$$= a^{k+1} + \sum_{i=1}^{k} \left[\binom{k}{i} + \binom{k}{i-1} \right] a^{k+1-i} b^i + b^{k+1}$$

Here we are applying Pascal's identity again.

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=1}^{k} \binom{k}{i-1} a^{k+1-i} b^i + b^{k+1}$$

Here we are again applying the rule $\sum (a_i + b_i) = \sum a_i + \sum b_i$.

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a(a^{k-i}) b^i + \sum_{i=0}^{k-1} \binom{k}{i} a^{k+1-(i+1)} b^{i+1} + b^{k+1}$$

Here we write a^{k+1-i} as $a(a^{k-i})$.

$$= \binom{k}{0} a^{k+1} b^0 + a \sum_{i=1}^{k} \binom{k}{i} a^{k-i} b^i + \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b(b^i) + \binom{k}{k} a^0 b^{k+1}$$

Here we write b^{i+1} as $b(b^i)$.

$$= \binom{k}{0} a(a^k) b^0 + a \sum_{i=1}^{k} \binom{k}{i} a^{k-i} b^i + b \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^i + \binom{k}{k} a^0 b(b^k)$$

Here we write b^{k+1} as $b(b^k)$.

$$= a \binom{k}{0} a^k b^0 + a \sum_{i=1}^{k} \binom{k}{i} a^{k-i} b^i + b \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^i + b \binom{k}{k} a^0 b^k$$

$$= a \left[\binom{k}{0} a^k b^0 + \sum_{i=1}^{k} \binom{k}{i} a^{k-i} b^i \right] + b \left[\sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^i + \binom{k}{k} a^0 b^k \right]$$

$$= a \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i + b \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

In a few places in this argument, we apply the rule $\sum ca_i = c \sum a_i$, which is a more compact form of $ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n)$

$$= a(a+b)^k + b(a+b)^k$$

$$= (a+b)^k (a+b)$$

$$= (a+b)^{k+1}$$

Therefore, we have shown that P_1 is true, ***and*** that if P_k is true, then P_{k+1} must also be true. By the principle of mathematical induction, this implies that P_n must be true for all $n \in \mathbb{N}$.

Homework

pp. 461-463 #2, 3, 5, 8, 11, 14, 15, 16, 17, 19