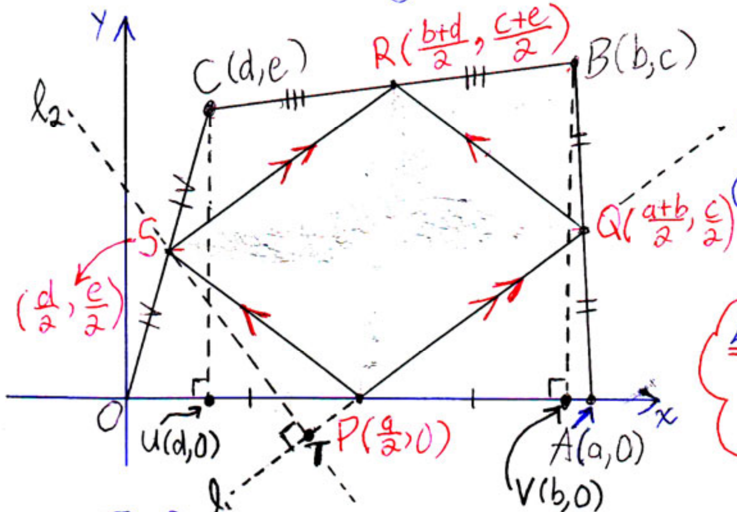
Varignon Parallelogram Investigation (Solutions)



Prove:

(a) PQRS is a parallelogram

(b) Area of quad OABC is twice the area of parallelogram PQRS

Legend: black, blue, green — given information; red → deduced information; pencil → or construction

Proof: Let P, Q, R and S be the midpoints of OA, AB, BC and CO respectively.

(a) Using the midpoint formula, the co-ordinates of P, Q, R and S are as shown in the diagram.

$$\therefore \text{slope } SR = \frac{\frac{c+e}{2} - \frac{e}{2}}{\frac{b+d}{2} - \frac{d}{2}} = \frac{\frac{c}{2}}{\frac{b}{2}} = \frac{c}{2} \times \frac{2}{b} = \frac{c}{b}$$

$$\text{and slope } PQ = \frac{\frac{c}{2} - 0}{\frac{a+b}{2} - \frac{a}{2}} = \frac{\frac{c}{2}}{\frac{b}{2}} = \frac{c}{b}$$

$\therefore$ slope SR = slope PQ

$\therefore$ $SR \parallel PQ$

Similarly, $PS \parallel QR$.

$\therefore$ PQRS is a parallelogram. //

(b) To calculate the area of parallelogram PQRS, take PQ as the base. Then extend QP to T in such a way that $ST \perp TQ$. Then ST is the height of parallelogram PQRS. Once we find the co-ordinates of T, we will be able to calculate the lengths of ST and PQ and hence, find the area of PQRS.

This is the CRUX of the problem.

(i) Find the equations of ℓ_1 and ℓ_2 (see diagram)

Slope of ℓ_1 = slope of $PQ = \frac{c}{b}$ (see part (a))

$\therefore$ slope of ℓ_2 = slope of $ST = \frac{-b}{c}$ ($\because PQ \perp ST$)

$\therefore$ by using the "slope = slope" method of finding equations of lines, an equation of ℓ_1 is $y = \frac{c}{b}x - \frac{ca}{2b}$ and an equation of ℓ_2 is $y = \frac{-b}{c}x + \frac{bd}{2c} + \frac{e}{2}$

At the point of intersection of ℓ_1 and ℓ_2,

$$y = y$$

$$\therefore \frac{c}{b}x - \frac{ca}{2b} = -\frac{b}{c}x + \frac{bd}{2c} + \frac{e}{2}$$

$$\therefore \frac{2bc}{1}\left(\frac{c}{b}x - \frac{ca}{2b}\right) = \frac{2bc}{1}\left(\frac{-b}{c}x + \frac{bd}{2c} + \frac{e}{2}\right)$$

$$\therefore 2c^2x - ac^2 = -2b^2x + b^2d + bce$$

$$\therefore 2c^2x + 2b^2x = b^2d + bce + ac^2$$

$$\therefore x(2c^2 + 2b^2) = b^2d + bce + ac^2$$

$$\therefore x = \frac{b^2d + bce + ac^2}{2(c^2+b^2)}$$

Substituting into the equation of ℓ_1, we obtain

$$y = \frac{c}{b}\left(\frac{b^2d + bce + ac^2}{2(c^2+b^2)}\right) - \frac{ca}{2b}$$

$$= \frac{c(b^2d + bce + ac^2) - ca(c^2+b^2)}{2b(c^2+b^2)}$$

$$= \frac{c[b^2d + bce + \cancel{ac^2} - \cancel{ac^2} - ab^2]}{2b(c^2+b^2)}$$

$$= \frac{\cancel{b}c(bd + ce - ab)}{2\cancel{b}(c^2+b^2)} = \frac{c(bd+ce-ab)}{2(c^2+b^2)}$$

$\therefore$ the co-ordinates of T are $\left(\frac{b^2d+bce+ac^2}{2(c^2+b^2)}, \frac{c(bd+ce-ab)}{2(c^2+b^2)} \right)$

Now we are ready to calculate the lengths of PQ and TS.

$$PQ = \sqrt{\left(\frac{a+b}{2}-\frac{a}{2}\right)^2 + \left(\frac{c}{2}-0\right)^2}$$
$$= \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2}$$
$$= \sqrt{\frac{b^2}{4}+\frac{c^2}{4}}$$
$$= \sqrt{\tfrac{1}{4}(b^2+c^2)}$$
$$= \tfrac{1}{2}\sqrt{b^2+c^2}$$

$$TS = \sqrt{\left[\frac{b^2d+bce+ac^2}{2(b^2+c^2)}-\frac{d}{2}\right]^2+\left[\frac{c(bd+ce-ab)}{2(b^2+c^2)}-\frac{e}{2}\right]^2}$$
$$= \sqrt{\left[\frac{b^2d+bce+ac^2-d(b^2+c^2)}{2(b^2+c^2)}\right]^2+\left[\frac{c(bd+ce-ab)-e(b^2+c^2)}{2(b^2+c^2)}\right]^2}$$
$$= \sqrt{\left[\frac{bce+ac^2-c^2d}{2(b^2+c^2)}\right]^2+\left[\frac{bcd-abc-b^2e}{2(b^2+c^2)}\right]^2}$$
$$= \sqrt{\left[\frac{c(be+ac-cd)}{2(b^2+c^2)}\right]^2+\left[\frac{b(cd-ac-be}{2(b^2+c^2)}\right]^2}$$
$$= \sqrt{\frac{c^2(be+ac-cd)^2}{4(b^2+c^2)^2}+\frac{b^2(cd-ac-be)^2}{4(b^2+c^2)^2}}$$
$$= \sqrt{\frac{c^2(be+ac-cd)^2+b^2[(-1)(be+ac-cd)]^2}{4(b^2+c^2)^2}}$$
$$= \sqrt{\frac{(be+ac-cd)^2\cancel{(b^2+c^2)}}{4(b^2+c^2)^{\cancel{2}}}}$$
$$= \frac{1}{2}(be+ac-cd)\sqrt{\frac{1}{b^2+c^2}}$$
$$= \frac{be+ac-cd}{2\sqrt{b^2+c^2}}$$

Technically, this should be $|be+ac-cd|$

$\therefore$ area parallelogram PQRS $= bh$
$$= PQ(ST)$$
$$= \left(\tfrac{1}{2}\cancel{\sqrt{b^2+c^2}}\right)\left(\frac{be+ac-cd}{2\cancel{\sqrt{b^2+c^2}}}\right)$$
$$= \tfrac{1}{4}(be+ac-cd)$$
$$= \tfrac{1}{4}(ac+be-cd)$$

Now refer again to the diagram on the first page

area quad OABC

$$= \text{area } \Delta OUC + \text{area } \Delta AVB + \text{area trapezoid } ABCU$$

$$= \tfrac{1}{2}de + \tfrac{1}{2}(a-b)c + \tfrac{1}{2}(e+c)(b-d)$$

$$= \tfrac{1}{2}de + \tfrac{1}{2}ac - \tfrac{1}{2}bc + \tfrac{1}{2}(eb-ed+cb-cd)$$

$$= \cancel{\tfrac{1}{2}de} + \tfrac{1}{2}ac - \cancel{\tfrac{1}{2}bc} + \tfrac{1}{2}be - \cancel{\tfrac{1}{2}de} + \cancel{\tfrac{1}{2}bc} - \tfrac{1}{2}cd$$

$$= \tfrac{1}{2}(ac+be-cd)$$

$$= 2\left[\tfrac{1}{4}(ac+be-cd)\right]$$

$$= 2(\text{area parallelogram } PQRS). //$$

Note:

1. Although we have managed to prove the result required for part (b), the method of proof is somewhat unsatisfying. It required a large number of algebraic gyrations. Later in the course we shall apply other methods to create far more elegant and satisfying proofs.

2. On the first page we identified the crux of the problem, which was to find the co-ordinates of T.

crux $\longrightarrow$ The basic, central or critical point or feature

In problem solving, identifying the crux of a problem is extremely important.